Less Room in n-Dimensions?
The Volume of an n-Dimensional Sphere Becomes Zero as $n \to \infty$

I still find this hard to believe. Somehow one thinks that there will be *more* room available in n-dimensions. But the unit sphere is circumscribed by the unit cube, so its volume is obviously less than 1 in all dimensions. The volume of a sphere of radius r in n = 2N or n = 2N+1 dimensions is easily shown to be,

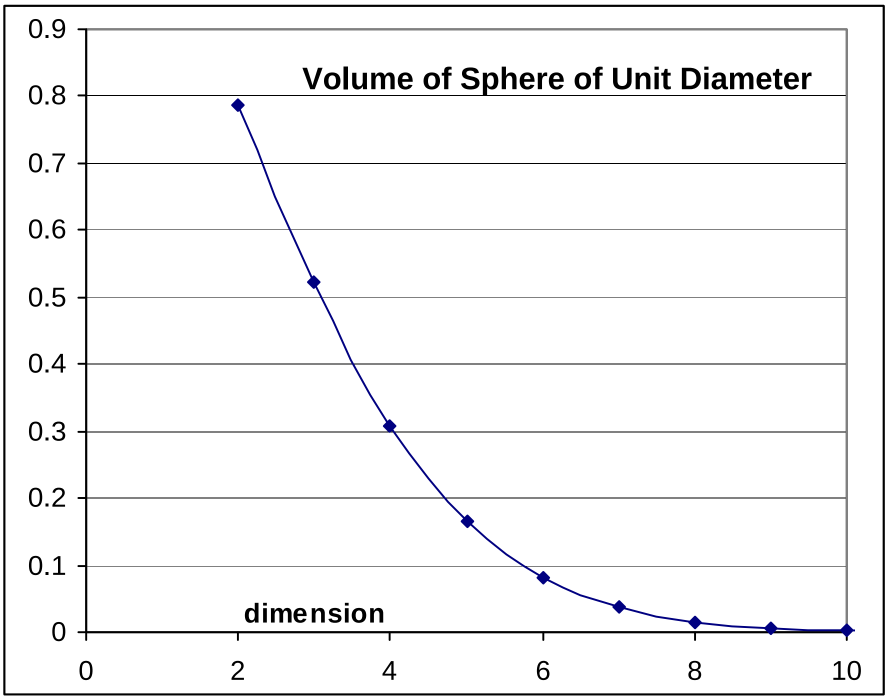
$$V_{2N} = \frac{\pi^N}{N!} r^{2N} \qquad \text{and} \qquad V_{2N+1} = \frac{2^{N+1}\pi^N}{(2N+1)!!} r^{2N+1} \tag{1}$$

Because of the factorials in the denominators, these volumes shrink to zero as the dimension becomes sufficiently large, even for arbitrarily large radii. Weird, isn't it?

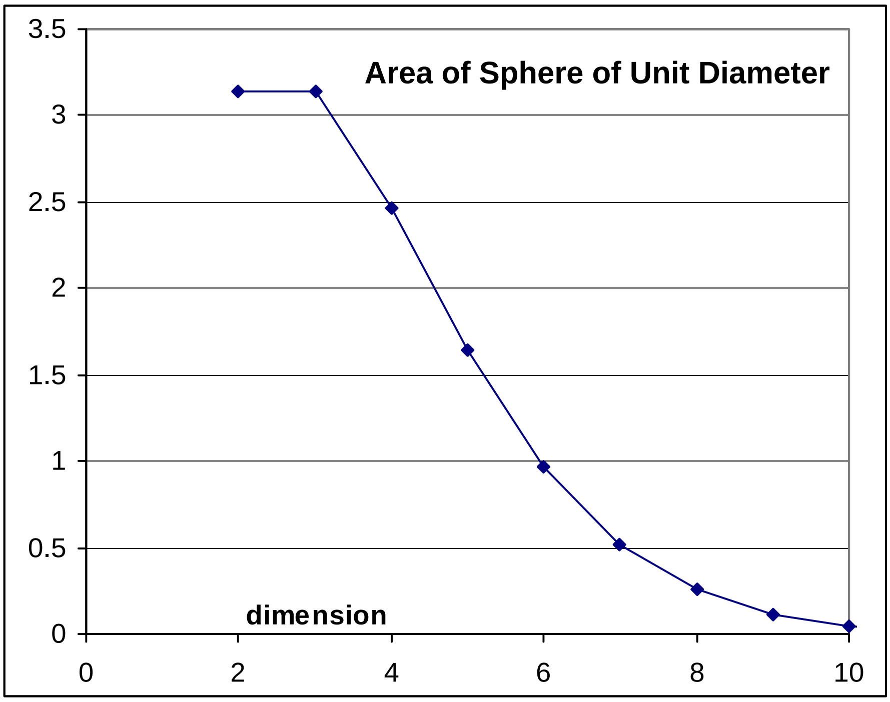
For spheres of unit diameter the volumes become,

$$V_{2N} = \frac{\pi^N}{2^{2N} N!} \qquad \text{and} \qquad V_{2N+1} = \frac{\pi^N}{2^N (2N+1)!!} \tag{2}$$

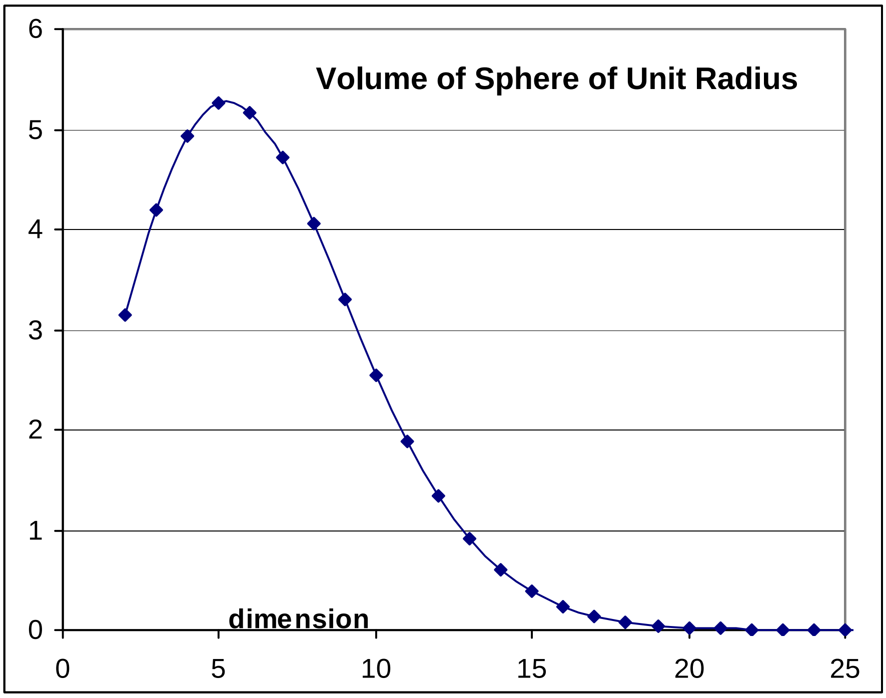
These are plotted against n below,

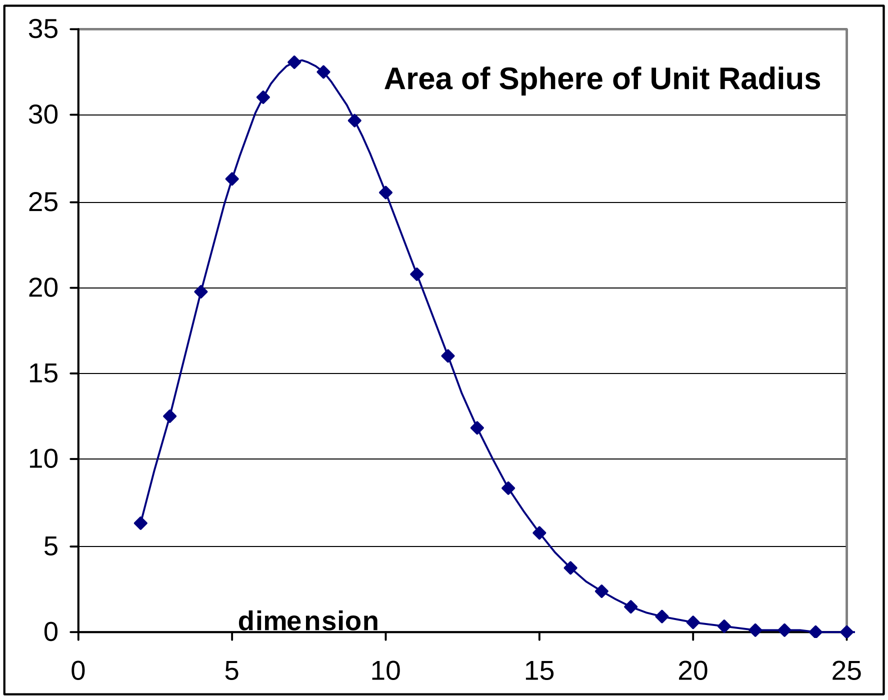


The corresponding surface area is simply A = nV/r, and is shown below for the sphere of unit diameter,



The striking thing is that the volume is greatest in 2-dimensions, whilst the surface area is equal greatest in 2-dimensions and 3-dimensions. This simple fact is sometimes obscured by plotting the volumes and areas of spheres with unit *radius*, as follows,





The volume of a sphere of unit radius is greatest in 5 dimensions, whereas the surface area of a sphere of unit radius is greatest in 7 dimensions. There is nothing fundamental about this, however, since different radii will give different values for the dimension which maximise the volume and area.

The fundamental result is the ratio of volume of the sphere to that of the circumscribed cube. For any size of sphere, this equals the volume of the sphere of unit diameter, as plotted initially. And this is greatest in 2D, with the area being equal greatest in 2D and 3D.

So, there is no more room available in higher dimensions.