The Relativity of Particles

The Fulling-Davies-Unruh effect is discussed and used as a poor man's approach to the Hawking radiance of black holes. Thermal radiation fields can arise in the vicinity of black holes, or simply due to the observer's acceleration. These effects are derived here in an accessible, if heuristic, manner. The opportunity is taken to review some of the basic features of quantum field theory. The generic lesson is that the presence or absence of particles is relative to the observer's state of motion, and hence to the spacetime geometry and the gravitational field.

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1. The Bogolubov Transformation

I stumbled upon Hawking's derivation of the radiance of black holes soon after he made the discovery, in the proceedings of a conference held at the Rutherford Laboratory, UK, in February 1974, see Hawking (1975). It took me years to understand it. However, the mechanism by which a spacetime can create a thermal radiation field is relatively simple to appreciate. This is illustrated here by the Fulling-Davies-Unruh (FDU) effect, Refs.[2-4], whereby an accelerated observer sees the Minkowski vacuum as a thermal field with a temperature proportional to the acceleration. The Hawking temperature of a black hole can be obtained from the FDU effect by substitution of the black hole's surface 'g'. This is not a rigorous derivation, of course, but it rationalises the result and makes the appearance of the Planck spectrum less mysterious.

Important though these results are in themselves, arguably more important is what they imply about the nature of particles in physics. We are accustomed to some measured quantities being different according to the state of motion of the observer. Relativity theory takes care of this, telling us how quantities transform between observers in a covariant manner – thus revealing the underlying objective nature of reality (at least in classical physics). But surely the presence of a particle is something that observers must agree about? Can one observer see a particle where another sees none? Surely a particle is a particle is a particle? But, no, this is not so. The presence or absence of particles can also be observer dependent. Actually we have seen the first intimations of this already in the conundrum of the radiation from a uniformly accelerating charge in classical electromagnetism (Chapter ?). The inertial and comoving observers do not agree regarding the presence of radiated photons. But it is in quantum field theory that the issue comes firmly to a head.

A transformation between observers means a spacetime transformation: observers differ in their position, orientation or state of motion. We do not have to look far to discover why spacetime might affect the concept of 'particle'. A particle is virtually defined as an eigenstate of the 4-momentum operator, $\hat{P}^{\mu}$. This is overtly a geometrical entity since it is the generator of isometries of the spacetime in question. It suffices to consider a scalar field to make the point. The quantum field theoretical description of a scalar field is,

$$\phi(x) = \sum_{\vec{k}} \left(u_{\vec{k}}(x) a_{\vec{k}} + u^*_{\vec{k}}(x) b^+_{\vec{k}} \right) \tag{1}$$

Here $u_{\vec{k}}$ are a complete set of positive frequency eigen-modes and $a_{\vec{k}}$ is the Fock-space particle annihilation operator. The second term in (1) consists of the corresponding negative frequency solution coupled with the creation operator for the anti-particle, $b^{+}_{\bar{k}}$. Both terms must appear if (1) is to respect the PCT theorem. In a curved spacetime we would require the existence of a globally timelike Killing vector field in order to define the 'time' direction and to enable a clear distinction between particles and anti-particles. The $u_{\bar{k}}$ are also orthonormal with respect to some inner product, (u,v), such that,

$$\left(u_{\bar{k}}, u_{\bar{k}'}\right) = \delta^3\left(\bar{k}' - \bar{k}\right) \quad \text{and} \quad \left(u_{\bar{k}}, u^*_{\bar{k}'}\right) = 0 \tag{2a}$$

where the inner product respects,

$$(u,v)^* \equiv (v,u) \equiv -\left(u^*, v^*\right) \tag{2b}$$

What does (1) look like from the perspective of a different observer? Denoting the second observer's quantities by a tilda, since the field is a scalar we must have,

$$\phi(x) = \sum_{\bar{k}} \left(\tilde{u}_{\bar{k}}(x)\tilde{a}_{\bar{k}} + \tilde{u}^*_{\bar{k}}(x)\tilde{b}^{+}_{\bar{k}}\right) \tag{3}$$

The second observer's basis functions, $\tilde{u}_{\bar{k}}$, are also orthonormal so the equivalent of (2) holds. But since the $\left\{u_{\bar{k}}, u^*_{\bar{k}}\right\}$ are a complete set of functions, the transformed functions must be expressible as,

$$\tilde{u}_{\bar{k}} = \sum_{\bar{k}'} \left(\alpha_{\bar{k}\bar{k}'} u_{\bar{k}'} + \beta_{\bar{k}\bar{k}'} u^*_{\bar{k}'}\right) \tag{4a}$$

$$u_{\bar{k}'} = \sum_{\bar{k}} \left(\alpha^*_{\bar{k}\bar{k}'} \tilde{u}_{\bar{k}} - \beta_{\bar{k}\bar{k}'} \tilde{u}^*_{\bar{k}}\right) \tag{4b}$$

The substitution of (4) in (3) and equating with (1), followed by appeal to (2), provides the transformation for the creation/annihilation operators,

$$\tilde{a}_{\bar{k}} = \sum_{\bar{k}'} \left(\alpha^*_{\bar{k}\bar{k}'} a_{\bar{k}'} - \beta^*_{\bar{k}\bar{k}'} b^{+}_{\bar{k}'}\right) \tag{5a}$$

$$a_{\bar{k}'} = \sum_{\bar{k}} \left(\alpha_{\bar{k}\bar{k}'} \tilde{a}_{\bar{k}} + \beta^*_{\bar{k}\bar{k}'} \tilde{b}^{+}_{\bar{k}}\right) \tag{5b}$$

Equs.(4,5) are known as Bogolubov transformations, Bogolubov (1958). The Bogoloubov coefficients follow from orthonormality and the properties of the inner product, (2a,b), giving,

$$\alpha_{\bar{k}\bar{k}'} = \left(\tilde{u}_{\bar{k}}, u_{\bar{k}'}\right) \qquad \text{and} \qquad \beta_{\bar{k}\bar{k}'} = -\left(\tilde{u}_{\bar{k}}, u^*_{\bar{k}'}\right) \tag{6}$$

Now the vacuum state as far as the first observer is concerned is such that,

$$a_{\bar{k}}\left|0\right\rangle = 0 \tag{7}$$

But for the second observer, acknowledging the possibility that they may not agree on the vacuum state, we have,

$$\tilde{a}_{\bar{k}}\left|\tilde{0}\right\rangle = 0 \tag{8}$$

In fact we see that we are obliged, in general, to regard the two vacuums as distinct since (5) gives us,

$$\tilde{a}_{\bar{k}}\left|0\right\rangle = \sum_{\bar{k}'} \beta^{*}_{\bar{k}\bar{k}'} b^{+}_{\bar{k}'}\left|0\right\rangle = \sum_{\bar{k}'} \beta^{*}_{\bar{k}\bar{k}'}\left|b,\bar{k}'\right\rangle \qquad (9)$$

where $b^{+}_{\bar{k}'}\left|0\right\rangle = \left|b,\bar{k}'\right\rangle$ represents an anti-particle of momentum $\bar{k}'$. Consequently the second observer sees the first observer's vacuum state as containing (anti)particles. There will also be particles, of course, derived in the same way using the conjugate field, ϕ^{+}.

The expectation value of the particle number operator $\tilde{a}^{+}_{\bar{k}}\tilde{a}_{\bar{k}}$ of the second observer in the $\bar{k}$ th mode with respect to the vacuum state of the first observer is thus,

$$\left\langle 0\left|\tilde{a}^{+}_{\bar{k}}\tilde{a}_{\bar{k}}\right|0\right\rangle = \sum_{\bar{k}'}\left|\beta_{\bar{k}\bar{k}'}\right|^{2} \qquad (10)$$

That is, when the first observer sees a vacuum the second observer sees $\sum_{\bar{k}'}\left|\beta_{\bar{k}\bar{k}'}\right|^{2}$ particles in the $\bar{k}$ th mode. The orthonormality of the basis functions $\{\tilde{u}_{\bar{k}}, \tilde{u}^{*}_{\bar{k}}\}$ is easily seen to be equivalent to the following relations between the Bogolubov coefficients, expressed in matrix notation,

$$(\alpha)(\beta)^{T} = (\beta)(\alpha)^{T} \qquad \text{and} \qquad (\alpha)(\alpha)^{+} - (\beta)(\beta)^{+} = I \qquad (11)$$

The diagonal components of $(\beta)(\beta)^{+}$ are just (10), so the observation of $\bar{k}$ th mode particles by the second observer in the vacuum state of the first observer is predicted if the corresponding diagonal component of $(\alpha)(\alpha)^{+}$ differs from unity.

Suppose now that the spacetime in question is Minkowskian and that both observers are inertial observers. The transformation between them is therefore a proper Lorentz transformation, consisting in general of an arbitrary rotation and an arbitrary boost (and a displacement of the spacetime origin if you wish). In this case the positive frequency basis functions $\{u_{\bar{k}}\}$ transform into combinations of only the positive frequency basis functions wrt the second observer, $\{\tilde{u}_{\bar{k}}\}$. Similarly, the negative frequency basis functions transform into combinations of the transformed negative frequency basis functions alone, $\{u^{*}_{\bar{k}}\} \to \{\tilde{u}^{*}_{\bar{k}}\}$. This can be seen most simply by considering the explicit plane wave basis,

$$u_{\bar{k}}(x) = \frac{1}{\sqrt{2\omega V}} e^{-ik\cdot x} \qquad (12a)$$

where $k\cdot x = \omega t - \bar{k}\cdot\bar{r}$ and the positive root is taken in $\omega = \sqrt{m^{2} + \left|\bar{k}\right|^{2}}$ and m is the mass of the field quanta (the particles in question). Note that we are using the usual particle physics convention here, dropping the factors of c and $\hbar$. The basis functions in the Lorentz transformed frame will just be,

$$\tilde{u}_{\bar{k}}(x) = \frac{1}{\sqrt{2\omega V}} e^{-i\tilde{k}\cdot x} \qquad (12b)$$

where $\tilde{k}$ is just the usual Lorentz transformation of the components of the 4-vector k. In this case it so happens that the transformed basis functions are trivial sums over the $\{u_{\bar{k}}, u^*_{\bar{k}}\}$ in the sense that $\tilde{u}_{\bar{k}}(x)$ equals one particular function from the set $\{u_{\bar{k}}\}$. However, the key thing is that (12b) does not require the negative frequency solutions, $u^*_{\bar{k}} \propto e^{+ik\cdot x}$. From (6) this means that all the Bogolubov β -coefficients are zero, and hence (9) shows that the vacuum seen by the two observers is the same. Two observers related by a Lorentz transformation will agree on the number of particles and the number of anti-particles. On the other hand, if the Bogolubov β -coefficients are *not* zero then the two observers will disagree in general regarding the number of particles present.

Before examining cases for which the Bogolubov β -coefficients are not zero we review very briefly some aspects of the basic formulation of quantum field theory.

2. Why Particle-Antiparticle Creation is Forced Upon Us

Using continuum-$\bar{k}$ basis functions, (12), what we have written as discrete sums in (1,3,4,5,9,10) will become integrals with the following integration measure,

$$\sum_k \to \frac{V}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3k \qquad (13)$$

Note that the arbitrary normalisation volume, *V*, cancels in quadratic integrals of the basis functions. The basis functions, (12), and hence the field, (1), are solutions of the relativistic field equation for scalar particles, i.e., the Klein-Gordon equation $(\partial^\mu \partial_\mu + m^2)\phi = 0$.

The identification of a suitable inner product caused some historical difficulties. The non-relativistic definition $(u,v) = \int u^* v \cdot d^3x$ is consistent under non-relativistic conditions with the interpretation of $|\phi|^2$ as a probability density, and hence with (u,u) being the total probability of a particle being found anywhere in space, i.e., unity. But the attempt to identify $|\phi|^2$ with a probability density in the relativistic case runs into three problems, all of them fatal. Firstly it does not transform under Lorentz transformations as a probability density should. Secondly it is not necessarily positive. And finally, its integral over all space is not constant, i.e., it is not conserved. In the relativistic context a conserved quantity is represented by a 4-vector current J^μ with vanishing 4-divergence, $\partial_\mu J^\mu = 0$ (see Chapter ?). The conserved quantity is then $\int J^0 d^3x$. A suitable current obeying $\partial_\mu J^\mu = 0$ is defined by $J^\mu = i\phi^+ \overleftrightarrow{\partial}^\mu \phi$ where the anti-symmetric derivative is defined by $u\overleftrightarrow{\partial}^\mu v \equiv u\partial^\mu v - (\partial^\mu u)v$. This transforms correctly under Lorentz transformations and produces a conserved quantity because $\partial_\mu J^\mu = 0$ by virtue of the fields obeying the Klein-Gordon equation[1]. The conserved quantity is therefore $(\phi,\phi) = i\int \phi^+ \overleftrightarrow{\partial}^t \phi \cdot d^3x$ where the inner product is defined as,

$$(u,v) = -i\int u\overleftrightarrow{\partial}_t v^+ d^3x \qquad (14)$$

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[1] Chapter ? shows how this expression for the conserved current can be deduced from Noether's Theorem for a complex field whose Lagrangian is invariant under a phase change $\phi \to e^{i\varepsilon}\phi$.

Some authors refer to this inner product as a "scalar product" but this is unfortunate since it is actually the time component of a 4-vector and hence not a scalar in the tensorial sense. However, we are still left with the problem of (ϕ,ϕ) defined by (14) not necessarily being positive. This can be seen if, in (1), we replace the creation and annihilation operators by ordinary functions of the momentum. In this case it is easily derived that,

$$(\phi,\phi)=\frac{V}{(2\pi)^3}\int_{-\infty}^{+\infty}\left[\left|a(\bar{k})\right|^2-\left|b(\bar{k})\right|^2\right]d^3k \tag{15a}$$

So if the anti-particle amplitudes are dominant, (15a) will be negative. In terms of the Fock space creation/annihilation operators (15a) becomes,

$$(\phi,\phi)=\frac{V}{(2\pi)^3}\int_{-\infty}^{+\infty}\left[a_{\bar{k}}^{+}a_{\bar{k}}^{-}-b_{\bar{k}}^{+}b_{\bar{k}}^{-}\right]d^3k\equiv\sum_{\bar{k}}\left[a_{\bar{k}}^{+}a_{\bar{k}}^{-}-b_{\bar{k}}^{+}b_{\bar{k}}^{-}\right] \tag{15b}$$

But $a_{\bar{k}}^{+}a_{\bar{k}}^{-}$ extracts the number of particles in mode $\bar{k}$ in a given state, i.e., $N_{\bar{k}}=\langle\psi|a_{\bar{k}}^{+}a_{\bar{k}}^{-}|\psi\rangle$, and similarly $b_{\bar{k}}^{+}b_{\bar{k}}^{-}$ extracts the number of anti-particles. So it now becomes physically acceptable that (15b) might be negative since this merely means that there are more anti-particles than particles.

Moreover, and most importantly, the fact that (15b) is conserved does not require that the number of particles and anti-particles are separately conserved – only their difference. This means that any number of particles and anti-particles may be created or annihilated as long as they are created or annihilated in pairs, so that the *difference* in their numbers is constant. Note that because particles and anti-particles are oppositely electrically charged, (15b) is equivalent to the conservation of charge.

It is possible, of course, that the number of particles will be conserved – provided that the number of anti-particles is also conserved. This is the case in classical, low energy situations. But in general, for example if sufficient energy is available, there is no reason to expect the number of particles to be conserved and it will not be.

Before ending this brief review of the basics of quantum field theory it is worth pointing out a crucial feature of the basis functions, (12), namely the appearance of the frequency in the normalising coefficient. Some authors are guilty of giving the impression that this is merely an arbitrary normalisation convention. Admittedly this factor need not be applied to the basis functions, it could appear instead in the integral measure in (13). But it *must* appear somewhere. The frequency in the denominator is essential so that, with the definition (14) for the inner product, the frequency term cancels in quadratic integrals. Otherwise the time derivative in (14) would result in such integrals having a factor of ω. In (15b), for example, we would be obliged to re-interpret the creation and annihilation operators so that the number operator was $\omega a_{\bar{k}}^{+}a_{\bar{k}}^{-}$. Alternatively expressed, with the usual interpretation of the creation/annihilation operators, the number of particles would be $N_{\bar{k}}=\langle\psi|a_{\bar{k}}^{+}a_{\bar{k}}^{-}|\psi\rangle/\omega$, so this denominator of ω will inevitably crop-up. It is clear that this frequency denominator must have an absolute significance, not be a mere convention, because it works through into the predictions for the energy dependence of experimental cross-sections. Its absolute nature can be traced back to the arguments made above for the adoption of (14) as the appropriate inner product.

3. Particle Creation by Uniform Acceleration

Let us now consider the case of a Minkowski space vacuum as seen by an observer undergoing constant acceleration (a Rindler observer, who we shall call R). Assuming this observer is instantaneously comoving with an inertial observer, S, at time zero in both frames then the velocity of R seen by S is $v = \tanh(g\tau)$, in units with $c = 1$, where τ is the accelerating observer's time coordinate. Here g is R's constant acceleration wrt the instantaneous comoving inertial frame. In contrast, wrt our fixed inertial frame, S, the acceleration is $a = g/\gamma_v^3$ where $\gamma_v = 1/\sqrt{1-v^2}$. Inevitably the acceleration seen by S is vanishingly small at early and late times since R's speed is then very close to the speed of light.

Suppose a plane wave of angular frequency ω and wavenumber k is propagating through Minkowski spacetime in the same direction as R for $\tau < 0$, i.e., the negative coordinate direction. We shall assume a massless scalar field for simplicity, so that $k = -\omega$. At his time τ, observer R will see this wave as Doppler shifted to the frequency,

$$\tilde{\omega} = \gamma_v(\omega - vk) = \frac{\omega(1+\tanh(g\tau))}{\sqrt{1-\tanh^2(g\tau)}} = \omega e^{g\tau} \tag{16}$$

Hence the wave is red-shifted to very low frequencies for $\tau < 0$, whereas for $\tau > 0$ it is blue-shifted to very high frequencies. (For future use we note that a wave propagating in the opposite direction has $\tilde{\omega} = \omega e^{-g\tau}$ and the red/blue shifting is reversed). Because R sees a time-varying frequency it must really comprise a spectrum of different frequencies in his frame of reference. The total phase at time τ in frame R is,

$$\phi(\tau) - \phi(0) = \int_0^\tau \tilde{\omega}\, d\tau = \int_0^\tau \omega e^{g\tau}\, d\tau = \frac{\omega}{g}\left(e^{g\tau} - 1\right) \tag{17}$$

which becomes simply $\phi(\tau) = \frac{\omega}{g} e^{g\tau}$ is we choose the reference zero phase appropriately. Consequently the wave seen by R is proportional to $\exp\left\{i\frac{\omega}{g}e^{g\tau}\right\}$. The frequency spectrum seen by R is the Fourier transform of this function wrt his time coordinate, τ, i.e.,

$$f(\Omega) = \int_{-\infty}^{+\infty} d\tau \cdot \exp\{i\Omega\tau\} \exp\left\{i\frac{\omega}{g}e^{g\tau}\right\} \tag{18}$$

where Ω is the (angular) frequency seen by R. Making the substitution $y = e^{g\tau}$, (18) becomes,

$$f(\Omega) = \frac{1}{g}\int_0^{+\infty} dy \cdot y^{(i\Omega/g - 1)} e^{i\omega y/g} \tag{19}$$

But recall that the gamma function is $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$, in terms of which (19) is readily evaluated to be,

$$f(\Omega)=\frac{1}{g}\left(\frac{g}{\omega}\right)^{i\Omega/g} e^{-\pi\Omega/2g}\cdot\Gamma(i\Omega/g) \qquad (20)$$

Now the frequency spectrum seen by R is just $S(\Omega)=|f(\Omega)|^2$. Hence, using the identity $\Gamma(iy)=\frac{\pi}{y\sinh(\pi y)}$ we get,

$$S(\Omega)=\frac{2\pi}{\Omega g}\cdot\frac{1}{\exp\{2\pi\Omega/g\}-1} \qquad (21)$$

The second factor in (21) is just a Planck factor, whose standard form is $(\exp\{\hbar\Omega/k_B T\}-1)^{-1}$, where $k_B = 1.38\times 10^{-23} J/K$ is Boltzmann's constant. This (almost) establishes that R will see thermal radiation with a temperature given by,

$$k_B T=\frac{\hbar g}{2\pi c} \qquad (22)$$

This is the Fulling-Davies-Unruh temperature seen by an observer accelerating at a rate g through Minkowski vacuum. In (22) we have re-introduced the speed of light, c, explicitly so that the reader may explore the magnitude of the temperature for various accelerations. For the acceleration due to gravity on Earth it is a mere 4×10^{-20} K – not great enough to upset low temperature physicists.

This simple derivation, which we have taken from Alsing and Milonni (2004), is probably the simplest achievable means of rationalising how the Planck factor $(\exp\{2\pi\Omega/g\}-1)^{-1}$ arises from accelerated motion. It takes much of the mystery away from the phenomenon since the Planck factor is seen to arise simply from the time-dependent Doppler shift due to the observer's acceleration.

Note that the frequency spectrum, (21), seen by R applies for whatever plane wave frequency ω we start with in frame S. Remarkably (21) is independent of ω. This is fortunate since the Minkowski vacuum in frame S consists, of course, of a superposition of all frequencies, as shown by (1). Had they corresponded to different spectra in frame R the combined effect might not be a Planck spectrum or have a well defined temperature.

The above simple, if heuristic, derivation has by-passed the need to calculate the Bogolubov coefficients. However, we noted previously that it would be the presence of negative frequencies (as seen by R for positive frequency waves in S) which would cause the Bogolubov coefficients $\beta_{\omega\omega'}$ to be non-zero, and hence to give rise to particle creation. That negative frequencies do indeed arise in the accelerated frame can be seem from (20) and (21), since $f(\Omega)$ is non-zero for $\Omega<0$.

The alert reader will have noticed, however, that (21) is ***not*** actually a Planck spectrum – because of the frequency dependent first factor. At this point we must note that the above 'derivation' is really just heuristic and we should not be surprised that it has gone slightly astray. For a better argued derivation we again turn to Alsing and Milonni (2004).

Firstly consider how the inertial observer would describe a thermal field. He would see each mode $\bar{k}$ as populated with $(\exp\{\hbar\omega/k_B T\}-1)^{-1}$ particles, i.e.,

$$\left\langle a_{\bar{k}}^{+} a_{\bar{k}'} \right\rangle = \frac{\delta^3\left(\bar{k}' - \bar{k}\right)}{\left(\exp\{\hbar\omega / k_B T\} - 1\right)} \tag{23}$$

We are not interested in spatial variations, so we consider the field at the origin only, so that (1), with the plane wave basis (12a), becomes,

$$\phi(t) = \sum_k \frac{1}{\sqrt{2\omega V}} \left(e^{-i\omega t} a_k + e^{+i\omega t} a_k^{+}\right) \tag{24}$$

Here we have assumed massless scalar particles which are equal to their own anti-particles and have reduced the problem to one spatial dimension only. The Fourier transformed field, which is the operator analogue of (18), is,

For $\Omega > 0$:
$$\hat{f}(\Omega) = \int_{-\infty}^{+\infty} dt \cdot \exp\{i\Omega t\} \cdot \phi(t) = \sum_k \frac{2\pi}{\sqrt{2\omega V}} a_k \delta(\omega - \Omega) \tag{25}$$

Using (23) the expectation value of the product $\hat{f}^{+}(\Omega) f(\Omega')$ in the thermal state is,

$$\begin{aligned} \left\langle \hat{f}^{+}(\Omega) f(\Omega') \right\rangle &= \left\langle \sum_k \frac{2\pi}{\sqrt{2\omega V}} a_k^{+} \delta(\omega - \Omega) \sum_{k'} \frac{2\pi}{\sqrt{2\omega' V}} a_{k'} \delta(\omega' - \Omega') \right\rangle \\ &= \sum_k \frac{2\pi^2}{\omega V} \delta(\omega - \Omega) \delta(\omega - \Omega') \frac{1}{\left(\exp\{\hbar\omega / k_B T\} - 1\right)} \\ &= \sum_k \frac{2\pi^2}{\Omega V} \delta(\omega - \Omega) \delta(\Omega - \Omega') \frac{1}{\left(\exp\{\hbar\Omega / k_B T\} - 1\right)} \end{aligned} \tag{26}$$

But $\sum_k \to \frac{V}{2\pi} \int dk$ for modes confined to one spatial direction so that, noting that there are two values of k for which $\omega = \Omega$, then we see that (26) becomes,

$$\left\langle \hat{f}^{+}(\Omega) f(\Omega') \right\rangle = \frac{2\pi}{\Omega} \cdot \frac{\delta(\Omega - \Omega')}{\left(\exp\{\hbar\Omega / k_B T\} - 1\right)} \tag{27}$$

This characterises the quantum field of a thermal state seen by an inertial observer. (Note that the numerical factor in (27) depends upon the normalisation convention for both the basis states and f).

Now let us investigate the quantum field seen by an accelerating observer when there is simply a vacuum in the inertial frame. The quantum field seen by the accelerating observer and expressed in terms of his own time coordinate, τ, is,

$$\phi(\tau) = \sum_k \frac{1}{\sqrt{2\omega V}} \left(\exp\left\{ i\varepsilon \frac{\omega}{g} e^{\varepsilon g \tau} \right\} a_k + \exp\left\{ -i\varepsilon \frac{\omega}{g} e^{\varepsilon g \tau} \right\} a_k^{+} \right) \tag{28}$$

where the sum is over all positive and negative wave-numbers, i.e., in both directions of propagation, and $\varepsilon = |k| / k$ in accord with the observation made after Equ.(16). The Fourier transformed field in analogy to (18) is,

$$f(\Omega) = \int_{-\infty}^{+\infty} d\tau \cdot \exp\{i\Omega\tau\} \sum_k \frac{1}{\sqrt{2\omega V}} \left(\exp\left\{ i\varepsilon \frac{\omega}{g} e^{\varepsilon g \tau} \right\} a_k + \exp\left\{ -i\varepsilon \frac{\omega}{g} e^{\varepsilon g \tau} \right\} a_k^{+} \right) \tag{29}$$

We will want to form the vacuum expectation value of the product $\hat{f}^{+}(\Omega) f(\Omega')$, so only the a_k^{+} will survive. The time integral is conducted in the same way as above, the main difference from (20) being that we have now accounted explicitly for the sum over all modes in (29). Thus we get, (30)

$$\left\langle \hat{f}^{+}(\Omega) f(\Omega') \right\rangle = \frac{1}{g^2} \sum_k \left(\frac{g}{\omega} \right)^{-i\varepsilon\Omega/g} e^{-\pi\Omega/2g} \Gamma(i\Omega/g)^* \cdot \sum_{k'} \left(\frac{g}{\omega'} \right)^{i\varepsilon\Omega'/g} e^{-\pi\Omega'/2g} \cdot \Gamma(i\Omega'/g) \cdot \frac{\delta(k'-k)}{2V\sqrt{\omega\omega'}}$$

$$= \frac{1}{2Vg^2} e^{-\pi(\Omega+\Omega')/2g} \Gamma(i\Omega/g)^* \Gamma(i\Omega'/g) \sum_k \frac{1}{\omega} \left(\frac{\omega}{g} \right)^{i\varepsilon(\Omega-\Omega')/g}$$

But it can be shown that $\sum_k \frac{1}{\omega} \left(\frac{\omega}{g} \right)^{i\varepsilon(\Omega-\Omega')/g} = 2Vg\delta(\Omega-\Omega')$ so that (30) becomes,

$$\left\langle \hat{f}^{+}(\Omega) f(\Omega') \right\rangle = \frac{1}{g} e^{-\pi\Omega/g} \left| \Gamma(i\Omega/g) \right|^2 \delta(\Omega-\Omega') = \frac{2\pi}{\Omega} \cdot \frac{\delta(\Omega-\Omega')}{\left(\exp\{\hbar\Omega/k_B T\} - 1 \right)} \tag{31}$$

where we have set the temperature as in (22). But (31) is identical to (27) thus demonstrating that a detector accelerating through Minkowski spacetime will behave just like an inertial observer's detector in a thermal field at temperature T. We also see from the derivation of (27) how the pre-factor $\propto 1/\Omega$ arises and that it is consistent with these observers detecting thermal radiation with a Planck spectrum.

4. Who Sees Particles?

It is uncontroversial that a detector carried by the accelerated observer will register thermal radiation at the temperature given by (22). In short-hand we say, "the accelerated observer sees particles". Since the starting assumption was that the inertial observer sees a Minkowski vacuum, that would appear to be the end of the matter – the inertial observer sees no particles. However that ignores the back-reaction of the accelerated particle detector on the field. If the accelerated observer carried no detector (in which case he would be 'blind' and could see nothing) then there would be no physical source which could possibly produce radiation to be seen by the inertial observer. But the existence of an accelerating physical device which can detect particles, and which is therefore necessarily coupled to the field, provides the potential for a back-reaction and the creation of radiation observable by the inertial observer, so-called "Unruh radiation". Whether this actually happens is contentious. One point of view, e.g., Birrell and Davies (1982), is that any transition of the accelerated detector must be accompanied by the appearance of a quantum in the field, $|0\rangle \to |1\rangle$, so that Unruh radiation will occur. They attribute the source of the energy which brings about the creation of this quantum, and the simultaneous excitation of the detector, to the agency which causes the detector to accelerate. On the other hand, some authors argue that Unruh radiation is not real. For example, Ford and O'Connell (2005) use an oscillator model of a particle detector and claim that this results in no Unruh radiation. The resolution of the conflict might lie in the nature of the detector. Perhaps Unruh radiation occurs for some types of detector but is not a necessary phenomenon for all possible detectors – but this is just speculation. Akhmedov and Singleton (2007) go as far as to claim that Unruh radiation has already been detected in circular motion (in the form of the depolarisation of electrons magnetically confined in storage rings).

5. Hawking Radiance of Black Holes

We now make a wild leap of faith that the Hawking temperature of black holes may be deduced from the Fulling-Davies-Unruh temperature, (22), by insertion of the gravitational acceleration at the event horizon. For a non-spinning Schwarzschild black hole the radius of the event horizon is $r_{bh} = 2GM / c^2$. Consequently the surface gravitational acceleration is $g = GM / r_{bh}^2 = c^4 / 4GM$. Inserting this in (22) gives the temperature of a non-spinning, uncharged black hole to be,

$$k_B T = \frac{\hbar c^3}{8\pi GM} \tag{32}$$

Hence, a solar mass black hole has a temperature of 6 x 10^{-8} K, whereas a black hole with a mass of 1 gram would have a temperature of 1.2 x 10^{26} K. A black hole of the Earth's mass, 6 x 10^{24} kg, would have a temperature of 0.02 K, nearly 18 orders of magnitude greater than the temperature due to the Earth's actual gravitational acceleration.

Of course we have not really derived (32), though it is the correct answer. In the context of a black hole, in contrast to the FDU effect, there is spacetime curvature and the predicted thermal radiation is seen by an inertial observer at rest with respect to the black hole at spatial infinity. However the result may be derived by computing the Bogolubov β -coefficients, (6), in a spherical wave basis. In the original Hawking (1975) approach, and as reiterated by DeWitt (1975), the key is to appreciate the importance of the actual formation of the black hole. Thus, at early times the matter which will eventually undergo gravitational collapse to from the black hole was still dispersed. This allows the definition of "IN" state basis functions, which play the role of the u_k functions in (6). However, at late times, after the black hole has formed, the basis states are modified by the intense gravitational field. These late basis functions, or "OUT" states, play the role of the $\tilde{u}_k$ functions in (6). This is rather like a scattering problem in which incoming spherical waves get scattered off the gravitational field. The phase of these scattered waves is non-linearly related to the asymptotic spacetime coordinates, just as happens in the FDU effect, Equ.(17). It is this which leads to the Bogolubov β -coefficients, (6), being non-zero. Carrying through this calculation one finds that the key integral is very similar to that of (18) and (29) used to derive the FDU effect, and the result is again a Planck spectrum. The detailed derivation shows that this Planck spectrum is seen, in the black hole case, filtered by a transmission coefficient.

What is the total power radiated by a black hole? For a single, massless scalar field the answer follows from the temperature, Stefan's law, and the surface area, giving,

Single massless scalar field: $$P = \frac{\hbar c^6}{15{,}360\pi (GM)^2} \tag{33}$$

The actual total radiated power depends upon how many fields actively participate, and upon their spin and mass. There are no massless scalar fields really, so (33) is not directly relevant. Moreover, higher spin suppresses the radiated power, so photons will radiate more power than gravitons, though both will contribute. If neutrinos were massless they would be the dominant form of radiation – being of lower spin than photons and also because there are more distinct types of neutrinos. However, the contribution of non-zero mass particles will be negligible for temperatures below their threshold, roughly for $k_B T < 2m$. Only upper bounds have been established for the neutrino masses. Assuming the lightest neutrino mass were 0.1 eV then a black hole would have to have a mass less than ~10^{20} kg, or less than ~10^{-5} Earth mass (less than ~10^{-11} solar masses) for neutrinos to contribute significantly to the radiation. For heavier black holes the radiation will be dominated by photons and the power will be roughly double that of Equ.(33) due to the two photon polarisation states.

As a completely tangential aside it is interesting to note the size of the numerical factor in (33). To those who have faith in dimensional analysis this is a salutary lesson. The actual radiated power differs from the dimensionally correct factor $\hbar c^6/(GM)^2$ by nearly five orders of magnitude.

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