The Newtonian Two-Body Problem: What is the Cone?

The Problem

In Newtonian gravity, consider two masses, m_1 and m_2, to be idealised as point-like. The general solution for the trajectory of one mass wrt the other is one of the conic sections: circle, ellipse, parabola, hyperbola (which includes a straight line through the point of impact as a degenerate case).

The Question

What is the cone?

The Equations of Motion

The position vectors wrt an inertial frame are $\bar{r}_1$ and $\bar{r}_2$. The position of 2 wrt 1 is $\bar{r} = \bar{r}_2 - \bar{r}_1$. The equations of motion are,

$$m_1\ddot{\bar{r}}_1 = G\frac{m_1 m_2}{r^2}\hat{r} \qquad \text{and} \qquad m_2\ddot{\bar{r}}_2 = -G\frac{m_1 m_2}{r^2}\hat{r} \tag{1a,b}$$

Defining the reduced mass by $\mu = \frac{m_1 m_2}{m_1 + m_2}$ we get an equivalent pair of simultaneous equations,

$$m_1\ddot{\bar{r}}_1 + m_2\ddot{\bar{r}}_2 = 0 \qquad \text{and} \qquad \mu\ddot{\bar{r}} = -G\frac{m_1 m_2}{r^2}\hat{r} \tag{2a,b}$$

Equ.(2a) says that the centre of mass moves at constant velocity. We can therefore adopted the CoM as the origin of our inertial frame of reference in which case,

$$m_1\bar{r}_1 + m_2\bar{r}_2 = 0 \tag{3}$$

Hence, if we can solve for $\bar{r} = \bar{r}_2 - \bar{r}_1$ the solutions for $\bar{r}_1$ and $\bar{r}_2$ follow immediately from (3).

In (2b) it is important not to make the error of equating $\ddot{\bar{r}}$ with $\ddot{r}\hat{r}$. Expressing the radial unit vector in Cartesian coordinates we have,

$$\hat{r} = \hat{x}cos\theta + \hat{y}sin\theta \tag{4a}$$

$$\dot{\hat{r}} = (-\hat{x}sin\theta + \hat{y}cos\theta)\dot{\theta} = \hat{\theta}\dot{\theta} \tag{4b}$$

$$\ddot{\hat{r}} = (-\hat{x}sin\theta + \hat{y}cos\theta)\ddot{\theta} - (\hat{x}cos\theta + \hat{y}sin\theta)\dot{\theta}^2 = \hat{\theta}\ddot{\theta} - \hat{r}\dot{\theta}^2 \tag{4c}$$

And $\ddot{\bar{r}} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\hat{r}}\right) = \ddot{r}\hat{r} + 2\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} = \ddot{r}\hat{r} + 2\dot{r}\hat{\theta}\dot{\theta} + r\left(\hat{\theta}\ddot{\theta} - \hat{r}\dot{\theta}^2\right)$ (5)

Substituting (5) into (2b) and equating terms proportional to $\hat{r}$ and $\hat{\theta}$ gives,

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -G\frac{m_1 m_2}{r^2} \qquad \text{and} \qquad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{6a,b}$$

Integrating (6b) gives $\dot{\theta} = A/r^2$ for some constant A which we chose to write as $A = L/\mu$ because we then have that the constant $L = \mu r^2\dot{\theta} = \mu r v_\theta$ where $v_\theta = r\dot{\theta}$ is the (relative) velocity in the θ direction which means that L is the angular momentum, which we conclude is a constant of the motion. In terms of the constant angular momentum the equation of motion for the radial coordinate, (6a), becomes,

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - G\frac{m_1 m_2}{r^2} \tag{7}$$

The first term on the RHS is the centrifugal force. This crucial term would be missing if we made the error of equating $\ddot{\bar{r}}$ with $\ddot{r}\hat{r}$ in (2b).

Derivation of the Equation of Motion from the Lagrangian

We have $PE = -G\frac{m_1 m_2}{r}$ and $KE = \frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2 = \frac{\mu}{2}\left|\dot{\vec{r}}\right|^2 = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$ (8)

So the Lagrangian is $\mathcal{L} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + G\frac{m_1 m_2}{r}$. The Euler-Lagrange equation for the angular coordinate, $\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)$ gives $\frac{d}{dt}\left(\mu r^2\dot{\theta}\right) = 0$, i.e., that the angular momentum, $L = \mu r^2\dot{\theta}$ is a constant of the motion.

The Euler-Lagrange equation for the radial coordinate, $\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right)$ gives,

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - G\frac{m_1 m_2}{r^2} = \frac{L^2}{\mu r^3} - G\frac{m_1 m_2}{r^2}$$

in agreement with (7). This illustrates the greater power of the Lagrangian method when working in non-Cartesian coordinates.

General Solution for the Orbits

The Integration Constants (Constants of the Motion)

As (7) is a second order ordinary DE for the function $r(t)$ a solution can be specified in terms of two constants, e.g., the initial values of position and radial velocity, $r(0)$ and $\dot{r}(0)$. The solution for $\theta(t)$ then follows from the integration of $\dot{\theta} = L/\mu r(t)^2$ in terms of one further integration constant, perhaps $\theta(0)$. However, since we may orient our coordinate system however we like, this latter integration constant can be set at will. There are therefore two constants which will define the complete solution. Rather than, say, $r(0)$ and $\dot{r}(0)$, the solution may be defined instead by the angular momentum, L, and the energy, E. The energy is given by,

$$E = KE + PE = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - G\frac{m_1 m_2}{r} \tag{9}$$

which we expect to be a constant. That $\dot{E}$ is indeed identically zero follows by substituting for $\dot{\theta} = L/\mu r(t)^2$ and then substituting for $\mu\ddot{r}$ from (7).

However, we are interested here only in the shape of the orbit, not the explicit time dependence of the vector position. In other words, we want to find r as a function of θ having eliminated the time parameter. From the above argument the general solution will follow if we can find the solution for an arbitrary L and E (subject to their being consistent).

The General Solution for the Orbit

It is proposed that the general solution for the orbit is,

$$r = \frac{P}{1 + e\cos\theta} \tag{10}$$

where P and e are constants and we have oriented the coordinates so that $\theta = 0$ corresponds to the closest approach of the orbit to the origin. (NB: e is not Euler's number, it turns out to be the eccentricity of the orbit). That (10) is a solution of (7) is shown as follows.

$$\dot{r} = \frac{Pe\sin\theta}{(1 + e\cos\theta)^2}\dot{\theta} = \frac{r^2}{P^2}Pe\sin\theta\frac{L}{\mu r^2} = \frac{Le\sin\theta}{\mu P}$$

Hence, $$\ddot{r} = \frac{Le\cos\theta}{\mu P}\dot{\theta} = \frac{L^2 e\cos\theta}{\mu^2 P r^2} = \frac{L^2}{\mu^2 P r^2}\left(\frac{P}{r} - 1\right) = \frac{L^2}{\mu^2 r^3} - \frac{L^2}{\mu^2 P r^2}$$

And this is identical to (7) iff, $$P = \frac{L^2}{G m_1 m_2 \mu} \tag{11}$$

It remains to find the constant e in terms of L and E. This is done by substituting for r and $\dot{r}$ in (9) and then substituting for θ using (10) at which point both r and θ vanish from the expression for E. Rearranging gives,

$$e = \sqrt{1 + \frac{2E}{\mu}\left(\frac{L}{Gm_1m_2}\right)^2} \tag{12}$$

- Hence $e < 1$ when $E < 0$, i.e., anticipating that e is the eccentricity, a bound orbit with negative energy has eccentricity less than 1, and we will see that it is an ellipse.
- Conversely, $e > 1$ when $E > 0$, i.e., an unbound "fly past" orbit which we will see is an hyperbola.
- The limiting case is $e = 1, E = 0$ which we will see is a parabola.
- The circular orbit has $e = 0$ and hence $\frac{2E}{\mu}\left(\frac{L}{Gm_1m_2}\right)^2 = -1$.

The Geometry of Orbits (10)

We need to show that (10) does indeed specify the conic sections with eccentricity e.

Ellipse

An ellipse whose centre is at the origin of Cartesian coordinates is given by,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{13}$$

Without loss of generality we can assume $a \geq b$. The eccentricity is then,

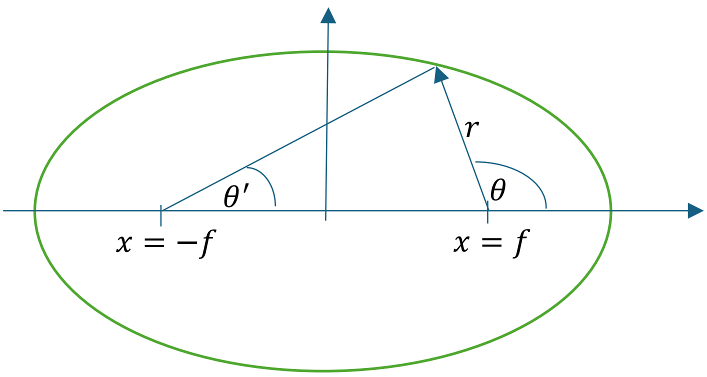
$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{14}$$

A circle has $a = b$, $e = 0$. An ellipse has $0 < e < 1$. The foci of the ellipse are at $x = \pm f$, $y = 0$ where,

$$f = ea = \sqrt{a^2 - b^2} \tag{15}$$

The foci are such that the sum of the distances from the two foci to any point on the ellipse is a constant (namely $2a$).

Recall that, in (10), the r, θ coordinates refer to the centre of mass. We now assume that we can identify the centre of mass with one of the foci, say $x = f$, $y = 0$, and show that that assumption leads to (13) being equivalent to (10).



Referring to the above diagram, the x coordinate can be expressed as $x = f + rcos\theta$. But we also have

$$r^2 = (f-x)^2 + y^2 = x^2 + f^2 - 2fx + b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= x^2\left(1 - \frac{b^2}{a^2}\right) - 2fx + b^2 + (a^2 - b^2) = e^2x^2 - 2eax + a^2 = (a - ex)^2$$

noting that $a > ex$. Hence $r = a - ex = a - e(f + rcos\theta) \Rightarrow r(1 + ecos\theta) = a - ef$

which is exact (10) with: $$P = a - ef > 0 \qquad (16)$$

This also establishes that the e in (10) is indeed the eccentricity. Note that (16) together with (11), (14), (15) establish one relationship between the two constants defining the ellipse, i.e., a, b and the constants of the motion, in this case L.

Had we used the focus at $x = -f$ we would have derived the same result in terms of the radial coordinate r' and the angle θ', i.e., $r'(1 + ecos\theta') = a - ef$.

The Hyperbola

A hyperbola may be defined by,

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \qquad (17)$$

The hyperbola has two disjoint branches, one at $x > 0$ and one symmetrically disposed at $x < 0$. The hyperbola (17) has planes of symmetry along the x axis and the y axis. The positive branch passes through the point $x = a$, $y = 0$ and this is the hyperbola's closest approach to the origin. For large x the hyperbola asymptotically approaches the straight lines defined by $y = \pm\frac{b}{a}x$ which provides the geometrical meaning of b (i.e., b is a times the gradient of the asymptote).

Just as an ellipse may be defined as the locus of points the sum of whose distances from the two foci is constant (namely $2a$) so an hyperbola can be defined as the locus of points the difference of whose distances from the two foci is constant (again $2a$). Because an ellipse is transformed into an hyperbola by the replacement $b \to ib$ then (15) suggests that the foci lie at $x = \pm f, y = 0$ where,

$$f = \sqrt{a^2 + b^2} \qquad (18)$$

This is confirmed by checking that the difference of the distances from the foci to any point on the hyperbola is always $2a$. The eccentricity is again defined so that $f = ea$, and hence we have $e > 1$ for an hyperbola.

Defining the r, θ coordinates from the focus at $x = f, y = 0$ we have,

$$r^2 = (x-f)^2 + y^2 = x^2 + f^2 - 2fx + b^2\left(\frac{x^2}{a^2} - 1\right)$$

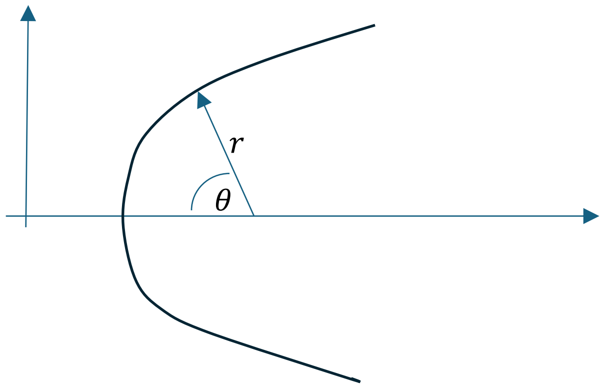
$$= x^2\left(\frac{b^2}{a^2} + 1\right) - 2fx - b^2 + (a^2 + b^2) = e^2x^2 - 2eax + a^2 = (ex - a)^2$$

noting that $ex > a$.

Hence $r = ex - a = -a + e(f + rcos\theta) \Rightarrow r(1 - ecos\theta) = ef - a$ which is again of the form (10) but now with,

$$P = ef - a > 0 \qquad (19)$$

The change of sign of the cosine term is merely because the angle has been defined as anticlockwise. Instead if we replace $\theta \to \pi - \theta$ we get exactly (10), $r(1 + ecos\theta) = P$ where we now have, for the positive branch,



Equs.(11) and (12) are again applicable and so we see that the hyperbola is indeed the solution for positive energy, i.e., $E > 0$ implies $e > 1$. Using (11) and (12) we can find the geometrical parameters a, b in terms of the dynamical constants of the motion, E, L. In a form applicable to both the ellipse and the hyperbola these are,

$$a = \frac{Gm_1m_2}{2|E|} \quad \text{and} \quad b^2 = \frac{L^2}{2\mu|E|} \tag{20}$$

The Cone

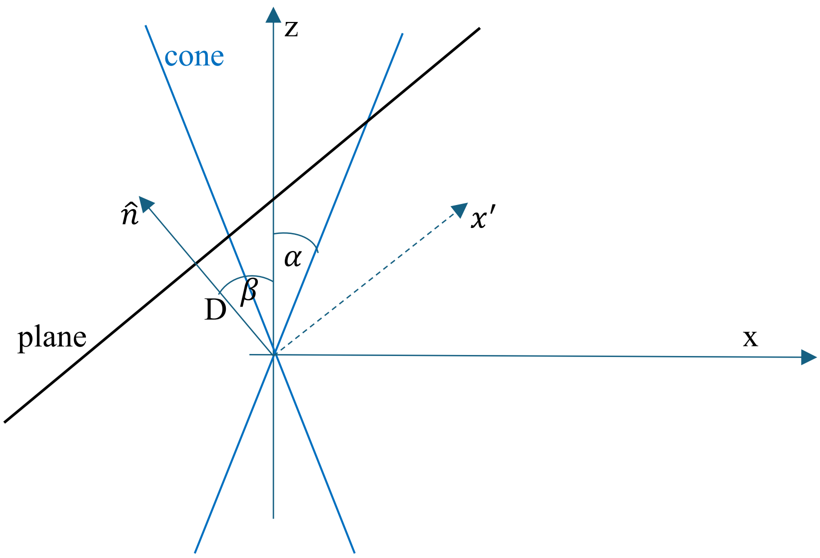
A cone is defined by its interior half-angle, α. If we place its axis along the z axis, with its apex at the origin, the equation of the cone is,

$$z = \pm\eta\sqrt{x^2 + y^2} \text{ where } \eta = cot\alpha \tag{21}$$

An arbitrary plane may be defined by its unit normal vector, $\hat{n}$, and its perpendicular distance from the origin, D (so the point on the plane nearest the origin is $D\hat{n}$). The equation of the plane is thus,

$$\hat{n} \cdot \bar{r} = D \tag{22}$$

Without loss of generality we can take $\hat{n}$ to lie in the x, z plane, so that $n_y = 0$, $n_x^2 + n_z^2 = 1$.



From which we see $n_z = cos\beta, n_x = -sin\beta$

The curve of intersection between the plane and the cone is thus,

$$z = \frac{D - n_x x}{n_z} = \pm\eta\sqrt{x^2 + y^2} \tag{23}$$

The unit vector $\hat{y}$ is parallel to the plane but $\hat{x}$ is not. Hence, to get a clear equation for the curve of intersection we need to transform the x coordinate to x' (see diagram). We have,

$$x' = x cos\beta + z sin\beta = n_z x - n_x z \tag{24}$$

But on the plane we have $n_x x + n_z z = D$ so $z = \frac{D - n_x x}{n_z}$. Substituting in (24) gives,

$$x = n_x D + n_z x' \tag{25}$$

Substituting (25) in (23) are re-arranging gives,

$$\eta^2 y^2 + (\eta^2 n_z^2 - n_x^2) x'^2 + 2 n_x n_z (1 + \eta^2) D x' + (\eta^2 n_x^2 - n_z^2) D^2 = 0 \tag{26}$$

By moving the origin of the x' coordinate we can transform to $\tilde{x} = x' - X$ to eliminate the linear term in this coordinate. This is achieved by the choice,

$$X = -\frac{n_x n_z (1 + \eta^2) D}{(\eta^2 n_z^2 - n_x^2)} \tag{27}$$

as may readily be checked. Equ.(26) then reduces to,

if $\eta^2 n_z^2 - n_x^2 > 0$ $\quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (ellipse) (28a)

if $\eta^2 n_z^2 - n_x^2 < 0$ $\quad \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ (hyperbola) (28b)

where, in both cases, we have,

$$a = \frac{\eta D}{|\eta^2 n_z^2 - n_x^2|} \quad \text{and} \quad b = \frac{D}{\sqrt{|\eta^2 n_z^2 - n_x^2|}} \tag{28c}$$

Note that the case $\eta^2 n_z^2 - n_x^2 = 0$ occurs when the plane is parallel to the side of the cone, i.e., $\beta = \frac{\pi}{2} - \alpha$ (giving a parabola, though we have not proved that here). Hence, the case $\eta^2 n_z^2 - n_x^2 > 0$ is when $\beta < \frac{\pi}{2} - \alpha$ and the plane intersects only the positive z part of the cone (ellipse). Conversely, when $\eta^2 n_z^2 - n_x^2 < 0$ then $\beta > \frac{\pi}{2} - \alpha$ and the plane intersects both the positive and negative parts of the cone (the two branches of the same hyperbola).

But what is the cone?

Equ.(20) defines a, b uniquely in terms of the constants of the motion, E, L. Equs.(28c) involve three unknown parameters: D, β and η, but there are only these two equations to be obeyed. Any set of the three parameters D, β and η which respect (28c), where a, b are given in terms of E, L by (20), will therefore provide a cone (η) and a plane (D, β) the intersection of which is the correct orbit. This means that…

> We may choose any cone we like (any angle α) and still be able to find a plane (defined by D, β) which correctly produces the orbit for any specified constants of the motion, E, L.

That there is a solution for D, β for any cone angle α follows by finding b/a from (28c) which yields the solution for β as,

$$n_z = cos\beta = \sqrt{\frac{1 \pm \left(\frac{b}{a}\right)^2 \eta^2}{1+\eta^2}} \tag{29}$$

where the + sign applies for the ellipse and the – sign for the hyperbola. Because $b < a$ the RHS of (29) is less than 1 and hence a (real) solution for angle β exists. The solution for D then follows from (28c) for any E, L.

As an example, consider the case of an ellipse with $b = a/2$. Solutions for β for the whole range of α between 0 and $\pi/2$ are,

α	$\eta = cot\alpha$	$n_z = cos\beta$	β
0.01	99.99667	0.707142	0.785348
0.1	9.966644	0.710622	0.780415
0.5	1.830488	0.784171	0.669437
1	0.642093	0.924141	0.392015
1.5	0.070915	0.998748	0.050040
1.56	0.010797	0.999971	0.007634

Note that we do not need to have a solution for α for any β, and indeed one can see from the above Table that there is no solution for α if we insist that $\beta > 0.8$ for an ellipse $b = a/2$. So we cannot choose the plane arbitrarily. But we can choose the cone arbitrarily as a solution for β always exists for any chosen α.