The Great Triangle Theorems: Part 1

I reprise the major triangle theorems largely as a prelude to tackling the last of the great triangle theorems, namely Morley's Theorem. It is salutary that the proof of results which have been familiar to me for over half a century did not always flow as fluently from my pen as I had expected.

I am concerned here with theorems which apply for arbitrary triangles – so Pythagoras will not appear. I have attempted to use the most elegant proofs in all cases. Morley's Theorem will follow in Part 2.

Contents

1. Notation and Terminology

$\bar{A}, \bar{B}, \bar{C}$ denote the position vectors of points A, B, C wrt an arbitrary origin.

A,B,C will be assumed to be non-colinear, hence there is a unique circle which passes through them, called the Circumcircle, the centre of which is the Circumcentre, denoted E.

$\bar{a}, \bar{b}, \bar{c}$ denote the position vectors of points A, B, C wrt the circumcentre, E.

Hence $a = b = c =$ the radius of the Circumcircle.

The length of the side opposite A is denoted $L_A = |\bar{C} - \bar{B}|$. Similarly L_B, L_C.

The three "altitudes" of a triangle are the lines drawn from each vertex which are perpendicular to the opposite side (extending the side if necessary). The altitude should also terminate on the opposite side (or its extension) so that the magnitude of the altitude vector equals the altitude as usually understood, and hence deserve the name.

The three medians of a triangle are the lines drawn from a vertex to the midpoint of the opposite side.

2. The Orthocentre, H

Define the Orthocentre, H, by its position vector wrt the Circumcentre, namely,

$$\bar{h} = \bar{a} + \bar{b} + \bar{c} \tag{1}$$

Theorem: The three altitudes of a triangle intersect at a common point and this point is H.

Proof:

The vector from C to H is $\overrightarrow{CH} = \bar{h} - \bar{c} = \bar{a} + \bar{b}$.

The vector from A to B is $\overrightarrow{AB} = \bar{b} - \bar{a}$.

Hence $\overrightarrow{CH} \cdot \overrightarrow{AB} = (\bar{a} + \bar{b}) \cdot (\bar{b} - \bar{a}) = b^2 - a^2 = 0$, because $a = b = c$.

Hence $\overrightarrow{CH}$ is perpendicular to side AB and, by construction, passes through C, and hence lies along the altitude at C (though not of the same magnitude as the altitude).

By symmetry, $\overrightarrow{AH}$ and $\overrightarrow{BH}$ will also lie along their respective altitudes (symmetry being a result od that of equ.(1)), and hence H is their common intersection. **QED**.

3. The Centroid, G

Theorem: The three medians of a triangle meet at a common point, called the Centroid, G.

Proof: The midpoint, M, of side AB is $\frac{(\bar{A}+\bar{B})}{2}$, because $\frac{(\bar{A}+\bar{B})}{2} - \bar{A} = \bar{B} - \frac{(\bar{A}+\bar{B})}{2}$.

Hence the median $\overrightarrow{CM} = \frac{(\bar{A}+\bar{B})}{2} - \bar{C}$.

Defining a point G which is 2/3 of the way along CM from C then

$$\overrightarrow{CG} = \frac{2}{3}\left(\frac{(\bar{A} + \bar{B})}{2} - \bar{C}\right) = \frac{(\bar{A} + \bar{B})}{3} - \frac{2}{3}\bar{C}$$

So the position vector of G is

$$\overrightarrow{CG} + \bar{C} = \frac{(\bar{A}+\bar{B})}{3} - \frac{2}{3}\bar{C} + \bar{C} = \frac{(\bar{A}+\bar{B}+\bar{C})}{3} \tag{2}$$

By the symmetry of this expression it follows that the same point G would be constructed as lying two-thirds away along the median from A and also two-thirds of the way along the median from B. Hence the medians have a common point of intersection, G, whose position vector of G is given by (2). **QED.**

4. E, G and H Are Colinear

Theorem: E, G and H are colinear.

Proof: This follows immediately because, if the Circumcentre is adopted as the origin than the position of G is given from (2) by $\overrightarrow{EF} = (\bar{a} + \bar{b} + \bar{c})/3$ which is proportional to (i.e., parallel to) $\overrightarrow{EH} = (\bar{a} + \bar{b} + \bar{c})$, from (1). **QED**.

5. The Circumcentre, E

Theorem: The Circumcentre, E, is the common point of intersection of the three perpendicular bisectors of the triangle's sides.

Proof: The equation of a line may be written $\bar{r} = \bar{u} + \lambda\bar{v}$ where $\bar{u}$ is an arbitrary point on the line and $\bar{v}$ is any vector parallel to the line. All points, $\bar{r}$, on the line may be expressed in this way for some λ whilst $\bar{u}$ and $\bar{v}$ remain fixed.

We know from §2 that $\overrightarrow{CH} = \bar{a} + \bar{b}$ is perpendicular to side AB, and we know from §3 that the midpoint of side AB is $\frac{(\bar{A}+\bar{B})}{2}$. Hence the equation of the perpendicular bisector of AB can be written $\bar{r} = \frac{(\bar{A}+\bar{B})}{2} + \lambda(\bar{a} + \bar{b})$. Shifting the origin of $\bar{r}$ to the Circumcentre, E, this can be written $\bar{r} = \frac{(\bar{a}+\bar{b})}{2} + \lambda(\bar{a} + \bar{b}) = \lambda'(\bar{a} + \bar{b})$ where $\lambda' = \lambda + \frac{1}{2}$.

In the same way the perpendicular bisector of side BC is $\bar{r} = \mu'(\bar{b} + \bar{c})$ and the perpendicular bisector of CA is $\bar{r} = \nu'(\bar{c} + \bar{a})$.

The three lines $\bar{r} = \lambda'(\bar{a} + \bar{b})$, $\bar{r} = \mu'(\bar{b} + \bar{c})$ and $\bar{r} = \nu'(\bar{c} + \bar{a})$ have a common point of intersection iff there are values of λ', μ', ν' such that all three give the same result for $\bar{r}$. Trivially there is such a solution, namely $\lambda' = \mu', = \nu' = 0$ which gives the common point of intersection of the perpendicular bisectors to be $\bar{r} = 0$. Recalling that we shifted the origin to the Circumcentre, this means that the Circumcentre is the common point of intersection of the three perpendicular bisectors. **QED**.

6. Coordinates of the Circumcentre

There is no neat vector formula for the position of the Circumcentre, but its Cartesian coordinates can be found as follows. The perpendicular bisector of side AB can be written $\bar{r} = \frac{1}{2}(\bar{A} + \bar{B}) + \lambda\bar{v}$ where $\bar{v}$ is perpendicular to $\bar{B} - \bar{A}$ and hence can be written $\bar{v} = (\Delta_y, -\Delta_x)$ where $\bar{B} - \bar{A} = (\Delta_x, \Delta_y)$. Writing the perpendicular bisector of another side in the corresponding way and solving for their intersection leads, after a little algebra, to,

$$E_x = \left[(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)\right]/D$$

$$E_y = -\left[\left(A_x^2 + A_y^2\right)(B_x - C_x) + \left(B_x^2 + B_y^2\right)(C_x - A_x) + \left(C_x^2 + C_y^2\right)(A_x - B_x)\right]/D$$

where $D = A_x\left(B_y - C_y\right) + B_x\left(C_y - A_y\right) + C_x\left(A_y - B_y\right)$

7. The Incentre

Theorem: The three bisectors of the angles at A, B and C intersect at a common point, I.

Proof: The unit vector along side AB, from A towards B, is $\frac{\bar{B}-\bar{A}}{L_C}$, and that from A to C is $\frac{\bar{C}-\bar{A}}{L_B}$. The direction of the bisector at A is thus the same as $\frac{\bar{B}-\bar{A}}{L_C} + \frac{\bar{C}-\bar{A}}{L_B}$ (though not normalised). Hence any point on the bisector of A is given by $\bar{r} = \bar{A} + \lambda\left(\frac{\bar{B}-\bar{A}}{L_C} + \frac{\bar{C}-\bar{A}}{L_B}\right)$. The bisector at B is therefore $\bar{r} = \bar{B} + \mu\left(\frac{\bar{A}-\bar{B}}{L_C} + \frac{\bar{C}-\bar{B}}{L_A}\right)$, and that at C is $\bar{r} = \bar{C} + \nu\left(\frac{\bar{A}-\bar{C}}{L_B} + \frac{\bar{B}-\bar{C}}{L_A}\right)$.

Equating for the intersection of the bisectors at A and B gives,

$$\bar{A} + \lambda\left(\frac{\bar{B}-\bar{A}}{L_C} + \frac{\bar{C}-\bar{A}}{L_B}\right) = \bar{B} + \mu\left(\frac{\bar{A}-\bar{B}}{L_C} + \frac{\bar{C}-\bar{B}}{L_A}\right)$$

Rearranging:

$$\bar{A} - \bar{B} + (\lambda+\mu)\frac{\bar{B}-\bar{A}}{L_C} = \left(1 - \frac{(\lambda+\mu)}{L_C}\right)(\bar{A}-\bar{B}) = +\mu\frac{\bar{C}-\bar{B}}{L_A} - \lambda\frac{\bar{C}-\bar{A}}{L_B}$$

For this to be possible we must have $\frac{\mu}{L_A} = \frac{\lambda}{L_B}$ and so,

$$\left(1 - \frac{(\lambda+\mu)}{L_C}\right)(\bar{A}-\bar{B}) = \frac{\mu}{L_A}(\bar{A}-\bar{B})$$

i.e., $$L_C L_A - L_A(\lambda+\mu) = L_C L_A - L_A\mu - L_B\mu = \mu L_C$$

so that, $$\mu = \frac{L_C L_A}{L_A+L_B+L_C} \qquad (3b)$$

gives the intersection point between the bisectors at A and B. By symmetry we immediately have that the same point is given also by,

$$\lambda = \frac{L_B L_C}{L_A+L_B+L_C} \qquad (3a)$$

But also by symmetry we will find that the bisector at C intersects with either that at A or that at B for,

$$\nu = \frac{L_A L_B}{L_A+L_B+L_C} \qquad (3c)$$

Hence there is a common intersection point given equivalently by (3a), (3b) or (3c). This defines the Incentre, I.

There is a more elegant proof, given in §12 below, but this requires some trigonometric formulae to be derived first.

8. The Inscribed Circle

Theorem: The Incentre is the centre of a circle which is tangential to all three sides of the triangle, and hence is the largest circle which fits inside the triangle.

Proof: Any point on the bisector at A has the same perpendicular distance from sides AB and AC, by symmetry. Any point on the bisector at B has the same perpendicular distance from sides BA and BC. Hence, the Incentre has the same perpendicular distance from all three sides. The circle of radius equal to this distance is therefore tangential to all three sides. **QED**.

9. The Cosine Rule

The familiar "Cosine Rule" follows immediately from the vector dot product. For our triangle with sides $\overrightarrow{AB} = \overline{b'}$, $\overrightarrow{AC} = \overline{c'}$ the third side is $\overrightarrow{BC} = \overline{c'} - \overline{b'}$. Hence the squared length of BC is $L_A^2 = \left|\overline{c'} - \overline{b'}\right|^2 = |\overline{c'}\,|^2 + \left|\overline{b'}\right|^2 - 2\overline{b'} \cdot \overline{c'} = L_B^2 + L_C^2 - 2L_B L_C cos(\theta_A)$ where θ_A is the angle at A, i.e., the angle between $\overline{b'}$ and $\overline{c'}$. This is the Cosine Rule which determines the length of one side in terms of the lengths of the other two and the cosine of the angle between them.

10. The Sine Rule

The area of the triangle is half the magnitude of the cross product between any two sides, i.e.,

Area = $\frac{1}{2}\left|\overline{b'} \times \overline{c'}\right| = \frac{1}{2}L_B L_C sin(\theta_A) = \frac{1}{2}L_C L_A sin(\theta_B) = \frac{1}{2}L_A L_B sin(\theta_C)$

Rearranging gives the familiar "Sine Rule":

$$\frac{L_A}{sin(\theta_A)} = \frac{L_B}{sin(\theta_B)} = \frac{L_C}{sin(\theta_C)} \quad (4)$$

where the side of length L_A is opposite the angle θ_A, etc.

11. The Bisector-Divider Rule

Referring to Figure 1, if BK bisects the angle FBC (each half being β) then the following equality between the ratios of sides applies,

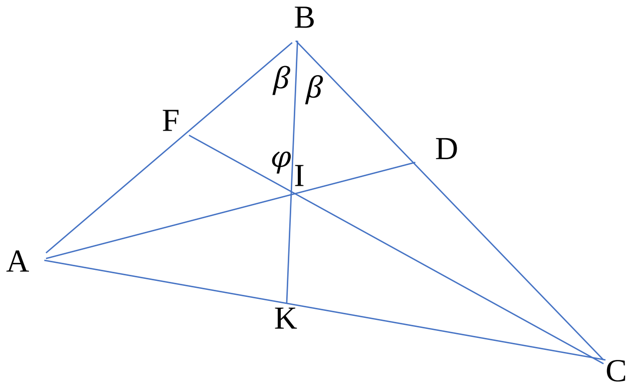
$$\frac{BC}{BF} = \frac{CI}{IF} \quad (5)$$

This can be deduced by applying the Sine Rule, (4), to triangles BIF and BCI giving,

$$\frac{FI}{sin\beta} = \frac{FB}{sin\varphi} \quad \text{and} \quad \frac{IC}{sin\beta} = \frac{BC}{sin(\pi-\varphi)} = \frac{BC}{sin\varphi}$$

which can be rearranged to give (5). Conversely, if (5) is known, then the dividing line must be a bisector.

Figure 1



12. The Incentre Revisited

Referring to Figure 1, in addition to (5) if we apply the same rule to triangles ACF, where AD is defined as the bisector of angle BAC, then we have also,

$$\frac{AC}{AF} = \frac{CI}{IF}$$

But this together with (5) gives $\frac{AC}{AF} = \frac{BC}{BF}$ which gives $\frac{AC}{BC} = \frac{AF}{BF}$ which establishes that CF is the bisector of angle ACB. That CF also passes through point I is established by §8. **QED**.

This is the more elegant proof of the Incentre, but it requires §10 and §11 to be established first.

13. Centre of Gravity

We now envisage a lamina, or plate, of uniform surface density, ρ, out of which some arbitrary shape is cut. (We will restrict to a triangle later).

The "Centre of Gravity" (cg) is that point on the shape such that the shape will balance on a knife edge passing through the point in any direction.

Here, "balance" means the net moment due to gravity is zero.

The moment about a given axis due to a small element of area δA is $\rho g D \delta A$ where D is the perpendicular distance from the axis.

Assume that we measure the position of the element of area, $\bar{r}$, from an origin lying on the axis at position $\bar{s}$, and suppose the direction of the axis is given by a unit vector $\hat{n}$. Then the total moment on the shape is,

$$\bar{M} = \rho g \int (\bar{r} \times \hat{n})\, dA = \rho g (\int \bar{r}\, dA) \times \hat{n} \quad (6)$$

To balance wrt a given orientation of knife-edge, $\hat{n}$, requires only that $\int \bar{r}\, dA$ is parallel to $\hat{n}$. However, if balance is to be possible for any $\hat{n}$, then we require $\int \bar{r}\, dA = 0$. This defines the location of the cg.

Theorem: If the net moment is zero about two non-colinear axes, then their intersection is the cg and the net moment is zero about any axis passing through that point.

Proof: Consider the intersection point of the two axes and wrt a Cartesian coordinate system whose origin is at that point define $M_x = \int x\, dA$ and $M_y = \int y\, dA$. If the first axis is at an angle θ wrt the x-axis then the perpendicular distance of the area element from this axis is $D = y cos\theta - x sin\vartheta$. Hence the moment about this axis is $M = M_x cos\theta + M_y sin\theta$. A similar expression holds for the second axis in terms of its orientation, θ', i.e., $M' = M_x cos\theta' + M_y sin\theta'$. Because $M' = M = 0$ we conclude that $M_x = M_y = 0$ and hence, as $M = M_x cos\theta + M_y sin\theta$ holds for any other axis, that the moment about any axis is zero. This establishes that the point of intersection of the initial two axes must be the cg. **QED**.

14. Theorem: For a Triangle, the Centroid is the Centre of Gravity

Proof: Consider initially an axis parallel to one side. If the triangle is oriented with its base along the x-axis then consider an axis parallel to the x-axis and at a height $y = Y/3$ above its base, where Y is the height of the triangle. (Hence, by (2), the axis passes through the centroid).

At any y position, the width of the triangle can be written as $L\left(1-\frac{y}{Y}\right)$ where L is the length of its base which lies at $y=0$. The moment about the axis is thus, (ignoring the constant ρg),

$$M=\int_0^Y \left(y-\frac{Y}{3}\right)dA$$

$$=L\int_0^Y \left(y-\frac{Y}{3}\right)\left(1-\frac{y}{Y}\right)dy=L\int_0^Y \left(\frac{4}{3}y-\frac{y^2}{Y}-\frac{Y}{3}\right)dy$$

$$=L\left(\frac{4}{3}\frac{Y^2}{2}-\frac{Y^2}{3}-\frac{Y^2}{3}\right)=0$$

Hence the cg does lie on the axis in question.

By repeating the same calculation for an axis parallel to one of the other sides, and one-third of the height wrt to that base, the total moment must again be zero, and hence their intersection, at $\frac{(\bar{A}+\bar{B}+\bar{C})}{3}$, must be the cg. So, the cg is the centroid. **QED**.

15. Moment of Inertia

We envisage either a shape cut from a uniform surface density plate, or, alternatively a prismatic body aligned with the z-axis whose x,y cross-section is constant.

Other than a constant facto equal to the material density times the thickness (or z-direction length) the moment of inertia about any axis lying in the x,y plane is the same as the second moment of area. Similar to (4) we defined,

$$I=\int|\bar{r}\times\hat{n}|^2\,dA \qquad (7)$$

where $\bar{r}$ is the position vector of the area element from any point lying on the axis in question (it makes no difference to *I* which point, as long as it lies on the axis).

Parallel Axis Theorem: The moment of inertia (second moment of area) about an axis which is a perpendicular distance s from the cg equals the moment of inertia about a parallel axis through the cg plus s^2A, where A is the total cross-section.

Proof: Define vector $\bar{s}$ to be perpendicular to the axis and of length s. Hence the vector to the area element wrt an origin on the axis in question is $\overline{r'}=\bar{r}-\bar{s}$, where $\bar{r}$ is the position of the same area element wrt to the parallel axis through the cg. Hence,

$$I=\int|\overline{r'}\times\hat{n}|^2\,dA=\int|(\bar{r}-\bar{s})\times\hat{n}|^2\,dA=\int(|\bar{r}\times\hat{n}|^2+s^2)\,dA-2(\bar{s}\times\hat{n})\cdot\int(\bar{r}\times\hat{n})\,dA$$

But the vector integral in the last term is zero because we are assuming that $\bar{r}$ is measured from the axis passing through the cg (see §13). Hence we get,

$$I=I_{cg}+s^2A$$

QED.

The Polar Moment of Inertia about a given point is the moment of inertia about the z-axis through that point and hence is given by $I_p = \int r^2 dA$, where r is the distance from the point to the area element.

Perpendicular Axis Theorem: The polar moment of inertia is equal to the sum of the in-plane moments of inertia about Cartesian directions x and y (irrespective of the orientation of the x,y coordinate system).

Proof: $I_p = \int r^2 dA = \int (x^2 + y^2) dA = I_x + I_Y$ **QED**.

Part 2 will address the last of the great triangle theorems, remarkably only discovered in 1899: Morley's Theorem.

Exercises for the Reader: Properties of the Orthocentre

These theorems have been taken from Orthocenter | Brilliant Math & Science Wiki. I invite you to attempt to prove them.

[1] Defining a new triangle from any two of the original vertices plus their Orthocentre, the Orthocentre of the new triangle is the third vertex.

[2] The reflection of the Orthocentre over any of the three sides lies on the Circumcircle of the triangle.

(A corollary of that is that, using a circular piece of paper and drawing an inscribed triangle on it, then after folding the paper inwards along the three edges the three arcs meet at the Orthocentre of the triangle).

[3] The angle ABC is supplementary to the angle AHC, i.e. they add to 180^o. This holds for all three angles, of course.

[4] The Circumcircle of the triangle formed by any two points of a triangle and its Orthocentre has the same radius as the circumcircle of the original triangle.

[5] If *any* point on the Circumcircle is reflected in turn over the three sides, resulting in three new points, these three points and the Orthocentre are collinear.

[6] The reflections of the triangle's altitudes over the angle bisectors intersect at the Circumcentre.