Tutorial Session 32 (T73S06) – Hysteresis Cycle Types

Last Update: 19/2/15

Relates to Knowledge & Skills items 1.14, 1.15, 1.16, 1.17, 1.18, 1.19

The concept of signed equivalent stress; Definition of equivalent strain range; Qualitative translation of load-time or temperature-time plots into stress-strain hysteresis cycles; Examples of different cycle types; Peak dwell, intermediate dwell; Interaction with smaller cycles; Crude methods for dwell stress estimation

Qu.: What equivalent stress is used in R5V2/3?

The Mises equivalent stress is used, usually denoted, $\bar{\sigma}$. It is calculated from the stress components using the usual von Mises formula,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)\right\}^{1/2} \quad (1)$$

Alternatively, in terms of principal stress components,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}^{1/2} \quad (2)$$

Qu.: What equivalent strain is used in R5V2/3?

The Mises equivalent strain is used, usually denoted, $\bar{\varepsilon}$. It is calculated from the strain components using the usual von Mises formula,

$$\bar{\varepsilon} = \frac{\sqrt{2}}{3}\left\{(\varepsilon_{11}-\varepsilon_{22})^2 + (\varepsilon_{22}-\varepsilon_{33})^2 + (\varepsilon_{33}-\varepsilon_{11})^2 + 6(\varepsilon_{12}^2+\varepsilon_{23}^2+\varepsilon_{31}^2)\right\}^{1/2} \quad (3)$$

Or, in terms of principal strain components,

$$\bar{\varepsilon} = \frac{\sqrt{2}}{3}\left\{(\varepsilon_1-\varepsilon_2)^2 + (\varepsilon_2-\varepsilon_3)^2 + (\varepsilon_3-\varepsilon_1)^2\right\}^{1/2} \quad (4)$$

Qu.: What is the shortcoming of this equivalent strain?

The equivalent strain defined by (3) or (4) applies only for incompressible behaviour, in other words for the plastic value for Poisson's ratio, $\nu = 0.5$.

Qu.: Why?

For elasticity, an equivalent strain can be defined simply from the usual elastic relation between stress and strain, that is $\bar{\varepsilon}^{el} = \bar{\sigma}/E$. It can be shown that the equivalent Mises strain defined in this way can be written in terms of the elastic strain components as,

$$\bar{\varepsilon}^{el} = \frac{1}{\sqrt{2}(1+\nu)}\left\{(\varepsilon_{11}-\varepsilon_{22})^2 + (\varepsilon_{22}-\varepsilon_{33})^2 + (\varepsilon_{33}-\varepsilon_{11})^2 + 6(\varepsilon_{12}^2+\varepsilon_{23}^2+\varepsilon_{31}^2)\right\}^{1/2} \quad (5)$$

In the plastic case, however, the effective Poisson's ratio is $\nu = ½$. Using this plastic value for ν, (5) reduces to (3). However, this means that neither expression is strictly correct if the elastic+plastic strain components are inserted into it.

R5 does not make use of the elastic equivalent strain, (5), preferring to use (3,4).

Qu.: What do these expressions give for uniaxial elastic stress ($\sigma_2 = \sigma_3 = 0$)?

The elastic equivalent strain, (5), gives,

uniaxial elastic stress: $\bar{\varepsilon}^{el}_{uniaxial} = \varepsilon_1$ (5b)

whereas the plastic expression used in R5 gives,

uniaxial elastic stress: $\bar{\varepsilon}^{pl}_{uniaxial} = \frac{2(1+\nu)}{3}\varepsilon_1$ (5c)

So, the R5 expression is not right under elastic conditions.

On the other hand, the elastic expression would not be right for plasticity - and it is plasticity which can produce the larger strains.

Qu.: Does this explain the use of $\bar{E}$ in the Ramberg-Osgood equations?

Yes.

In Session 31 we saw that the Ramberg-Osgood equations used $\bar{E} = \frac{3}{2(1+\nu)}E$ rather than E. This factor is just the ratio of the numerical factors in (3) and (5) and corrects the elastic part of the Ramberg-Osgood strain to be consistent with the R5 definition of equivalent strain, Equ.(3). By this ruse the equivalent strain is always correct.

Qu.: What strain range is used in fatigue?

R5V2/3 Appendix A10, Section A10.2.1 discusses the Tresca and Rankine equivalent strains in the context of evaluating the fatigue damage. This will be dealt with in detail in Session 35. In practice I suspect that the Mises equivalent is generally used since this is the strain range which falls out from the hysteresis cycle construction, and Appendix A10 does not give sufficient advice on how to calculate the Rankine strain. However in Session 35 I show this may be done simply.

Qu.: What strain range is used in cycle construction

The Mises strain range is used to construct hysteresis cycles according to R5V2/3 Appendix A7 [noting enhancements to be discussed in Session 33, e.g., R5V2/3, Appendix A7, Equ.(A7.4)].

The important thing to note is that the Mises strain range is defined via the ranges of the components, not as the difference of two Mises strains. Hence, if the change of the strain components between two times are as follows,

$$\Delta\varepsilon_{ij}(t,t') = \varepsilon_{ij}(t) - \varepsilon_{ij}(t') \text{ for all i, j} \quad (6)$$

then the Mises equivalent strain range is defined by, (7)

$$\Delta\bar{\varepsilon}(t,t') = \frac{\sqrt{2}}{3}\{[\Delta\varepsilon_{11}(t,t') - \Delta\varepsilon_{22}(t,t')]^2 + [\Delta\varepsilon_{22}(t,t') - \Delta\varepsilon_{33}(t,t')]^2 + [\Delta\varepsilon_{33}(t,t') - \Delta\varepsilon_{11}(t,t')]^2 + 6[\Delta\varepsilon_{12}^2(t,t') + \Delta\varepsilon_{23}^2(t,t') + \Delta\varepsilon_{31}^2(t,t')]\}^{1/2}$$

(as opposed to $\bar{\varepsilon}(t) - \bar{\varepsilon}(t')$). The strain range for the cycle is then identified as the maximum value of the equivalent strain change between any two instants of time during the cycle.

Qu.: How is the Mises equivalent stress range defined in R5?

Like the strain range, the Mises equivalent stress range is defined by taking the Mises combination of the ranges of the stress components,

$$\Delta\bar{\sigma} = \frac{1}{\sqrt{2}}\left\{(\Delta\sigma_{11} - \Delta\sigma_{22})^2 + (\Delta\sigma_{22} - \Delta\sigma_{33})^2 + (\Delta\sigma_{33} - \Delta\sigma_{11})^2 + 6(\Delta\sigma_{12}^2 + \Delta\sigma_{23}^2 + \Delta\sigma_{31}^2)\right\}^{1/2} \quad (8)$$

(see R5V2/3 Appendix A2, §A2.3.4).

Qu.: Does this mean that $\Delta\bar{\sigma}_{AB}$ differs from $\bar{\sigma}_B - \bar{\sigma}_A$?

Yes.

Numerical examples of this are given below.

Note that you would get into a great deal of difficulty if you tried to define absolute Mises stresses under specific conditions (say condition B) by adding (or subtracting) the Mises stress range $\Delta\bar{\sigma}_{AB}$, defined by (8), to an absolute stress under some previous conditions (A) by calculating $\bar{\sigma}_A$ from its stress components. This would give you a quantity $\bar{\sigma}_B^{R5} = \bar{\sigma}_A \pm \Delta\bar{\sigma}_{AB}$ which would differ (sometimes radically) from $\bar{\sigma}_B$ calculated directly from the stress components under condition B.

It is important to realise that this is not how the crack initiation assessment in R5V2/3 works. Instead the R5V2/3 initiation assessment depends only upon,

- The Mises equivalent stress ranges, as defined by (8), and,
- Estimating the absolute stresses by "symmetrising" the hysteresis cycle. Exactly what is meant by "symmetrising" the hysteresis cycle will be defined in Session 33. For now note that it ***roughly*** means putting the centre of the cycle at zero stress.

Qu.: Why does R5V2/3 use this definition of Mises stress range?

It is to enable an assessment methodology based on a single stress measure, essentially a signed equivalent stress. How would you carry out an assessment using all the stress components? What cyclic stress-strain curve would you use?

Qu.: But why not just use the Mises stress under each separate loading condition?

This is answered by the illustrations of Table 1. For example, the first illustration involves as stress state of (200, 0, 0) becoming a stress state of (0, 200, 0) under some change of conditions. But both these stress states have the same equivalent stress, i.e., 200. So $\bar{\sigma}_B - \bar{\sigma}_A$ in this case is zero. But clearly there ***is*** a change of stress, and it is likely to be damaging. Basing an assessment on $\bar{\sigma}_B - \bar{\sigma}_A$ in this case would obviously be wrong.

R5V2/3 would assess this case using an equivalent stress range of 346 MPa. With perfect cycle symmetrisation this would be interpreted as a hysteresis cycle between -173 MPa and +173 MPa. (I'm interpreting these stresses as elastic-plastic stresses for the purposes of this illustration).

The R5V2/3 initiation assessment would then be identical to a uniaxial case for which the stress changed from (-173, 0, 0) to (+173, 0, 0). Just how accurate is it to assume that the actual stress change, from (200, 0, 0) to (0, 200, 0) is equivalent in terms of damage to a cycle from (-173, 0, 0) to (+173, 0, 0)? I haven't a clue. This is one of the many aspects of R5 which has received little or no experimental validation. But at least the procedure does provide a sensible basis for assessment – whereas the use of $\bar{\sigma}_B - \bar{\sigma}_A$ clearly does not.

Note the considerable difference under R5V2/3 rules between the cycle from (200, 0, 0) to (0, 200, 0), which gives $\Delta\bar{\sigma} = 346$, and a cycle from (200, 0, 0) to (0, -200, 0), which gives $\Delta\bar{\sigma} = 200$. Do these two types of cycle really differ so much in terms of their damage-creating potential? I don't know, but it is quite possible.

> R5V2/3 assessments can be very sensitive to the absolute stress level predicted for the creep dwell, so these illustrations demonstrate how dependent the result may be on the specific triaxial procedure assumed in R5, with little validation.

Table 1: Illustrations of Calculation of Mises Equivalent Stress Range Using the R5 Definition, Equ.(8), Contrasted With the Difference of the Mises Stresses
(The stress components given below can be interpreted as the hoop, axial and radial stresses for some axisymmetric structure, with shear components assumed zero in these illustrations).

Stress Components Condition A	$\bar{\sigma}_A$	**Stress Components Condition B**	$\bar{\sigma}_B$	**Ranges of components (B – A)**	$\Delta\bar{\sigma}_{AB}$ **using Equ.(8)**	**Compare with** $\lvert\bar{\sigma}_A - \bar{\sigma}_B\rvert$
(200, 0, 0)	200	(0, 200, 0)	200	(-200, 200, 0)	346	0
(200, 0, 0)	200	(0, -200, 0)	200	(-200, -200, 0)	200	0
(200, 0, 100)	173	(0, -200, 0)	200	(-200, -200, -100)	100	27
(200, 50, -70)	234	(100, 70, 100)	30	(-100, 20, 170)	234	204
(100, 50, -170)	249	(50, 20, -50)	89	(-50, -30, 120)	161	160

Qu.: What are "signed equivalent" stresses in R5?

In normal engineering practice, equivalent stresses and strains are defined to always be positive.

However the R5V2/3 hysteresis cycle construction procedure automatically leads to about half the cycle being on the compressive side – despite Mises equivalent stress ranges having been used. The points on the hysteresis cycle are therefore "signed equivalent stresses".

These are defined by giving the Mises stress a sign equal to the sign of the principal stress of greatest magnitude. Hence, determine which of $\lvert\sigma_1\rvert, \lvert\sigma_2\rvert, \lvert\sigma_3\rvert$ is the largest, say $\lvert\sigma_K\rvert$ and set,

$$\text{signed Mises} = \text{sign}(\sigma_K) \text{ x Mises stress}$$

Actually it is never necessary to use this definition when carrying out an R5V2/3 assessment because the hysteresis loop construction does not actually require it. However, you need this concept if using FEA to construct the cycle...

Qu.: Plotting hysteresis cycles in FEA

If you carry out an elastic-plastic finite element analysis involving load cycles, you might very well want to plot the resulting hysteresis cycle. (In fact you really should). But at this point you are brought hard up against the problem. There are no 'equivalent' quantities to use which will give you a nice continuous hysteresis loop. The signed Mises stress defined as above will produce a discontinuous loop in general.

If all stress components change in proportion as the loads are varied (proportional loading) then there is no problem. The hysteresis loops will then be smooth and continuous and sensible looking using the definition of signed Mises stress given above.

However, proportional loading will not prevail in most realistic cases. Consider a reducing load so that the maximum principal stress is falling. There will generally come at time at which the principal stress of greatest magnitude switches from one component to another, which may be of the opposite sign. So the signed equivalent stress will typically change discontinuously from some non-zero positive value to an equal and opposite negative value. A hysteresis loop plotted from FE results will generally have its middle section missing.

This problem is alluded to in R5V2/3 Appendix A3 but without resolution. I don't know if there is any agreed procedure to use in this circumstance. Does anyone else know? I guess one has to make do with hysteresis loops which consist only of the upper and lower portions with a gap between them.

Qu.: Does the same problem affect "signed equivalent strain"?

Yes.

When plasticity or creep is occurring, the equivalent strain in FE programs like ABAQUS is defined incrementally, i.e., $\bar{\varepsilon} \equiv \int d\bar{\varepsilon}$. Using this definition each equivalent strain increment is positive, so the total equivalent strain can only increase monotonically irrespective of whether yielding is occurring in tension or compression. So you cannot plot a hysteresis loop using this type of equivalent strain. (I think this is what ABAQUS gives you if you ask for PEEQ - but don't take my word for it).

So you need to use a "signed equivalent strain increment" to get a sensible hysteresis loop. This involves the same sort of problems of discontinuity as for the signed equivalent stress. I don't know if ABAQUS will output such a quantity. But you can get at all the strain components so you should be able to calculate whatever you wish in a subroutine. Has anyone done this? Remember if you are doing an elastic-plastic FE analysis with cyclic loads then you really should plot the hysteresis loop. It's the hysteresis loop that really tells you what's going on.

Qu.: Does the discontinuity of signed equivalents cause problems in R5?

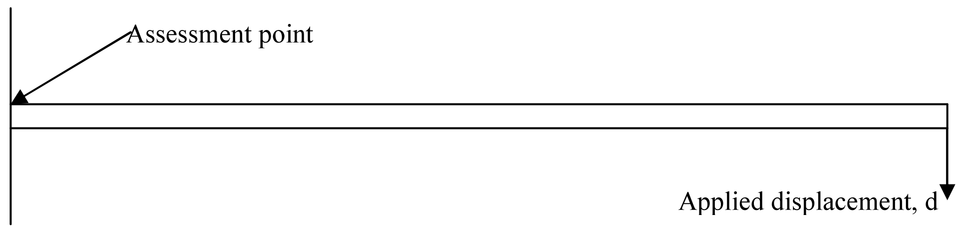
No.

Although I refer loosely to the R5V2/3 initiation assessment being based upon "hysteresis cycle construction" really we just construct a few points on the cycle: the top and bottom peaks, and perhaps some other transient stress peak points, plus the points representing the start and the end of the creep dwell. So we are never bothered about the whole hysteresis loop – and hence any discontinuity at small stresses goes unnoticed.

Qualitative Hysteresis Cycle Construction

In all examples it is assumed that $K_S S_y$ is temperature independent for simplicity

Figure 1: Displacement Loaded Cantilever



Case 1 Loading

The displacement, d, would cause an elastic stress >2 $K_S S_y$

(a) Apply the displacement, d;

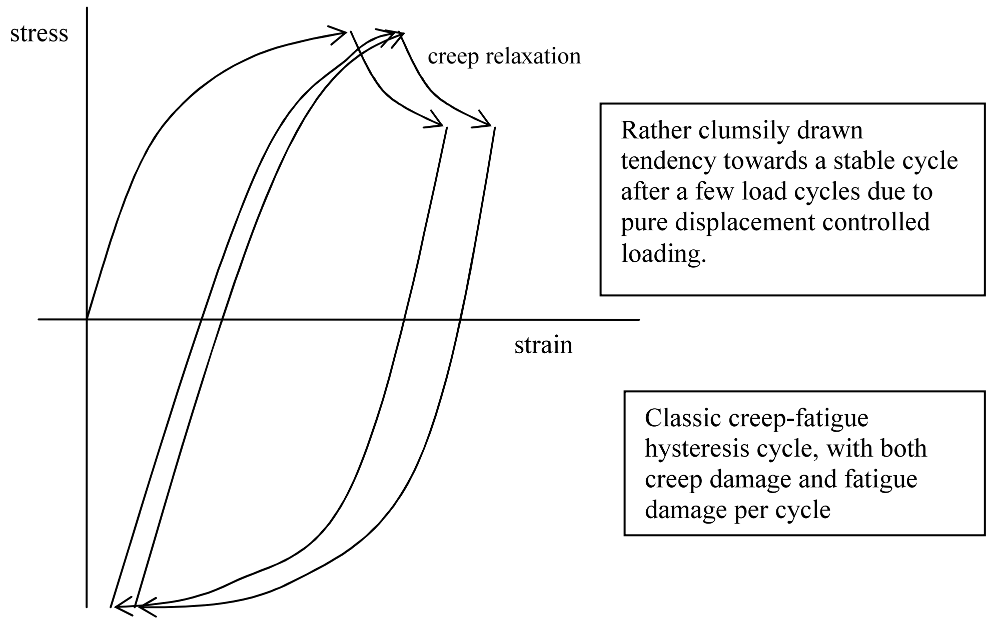
(b) Hold the displacement at high temperature;

(c) Apply the reverse displacement (-d);

(d) Repeat from (a).

Qu.: What does the stress-strain trajectory look like?

Figure 2: Hysteresis Loop for Case 1



Qu.: What would the stress-strain trajectory look like if the applied displacement were replaced by an applied load?

Case 2 Loading

(e) Apply the load, F;

(f) Hold the load constant at high temperature;

(g) Remove the load;

(h) Repeat from (a).

Figure 3a: Hysteresis Loop for Case 2 (modest F, no ratcheting)

This loop would apply if, for example, there were a stress concentrating feature at the assessment point so that the bulk of the cantilever remained elastic. The hysteresis loop applies to the local stresses and strains, and therefore the stress appears to 'relax' during creep due to redistribution. Reverse plasticity on unloading can result in a closed, non-ratcheting, hysteresis loop.

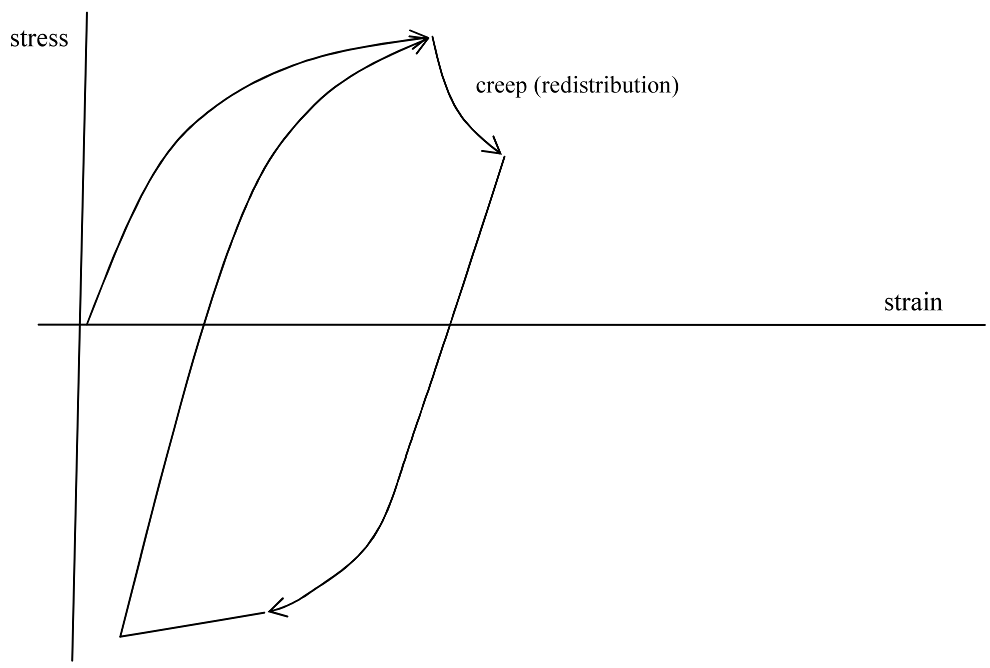
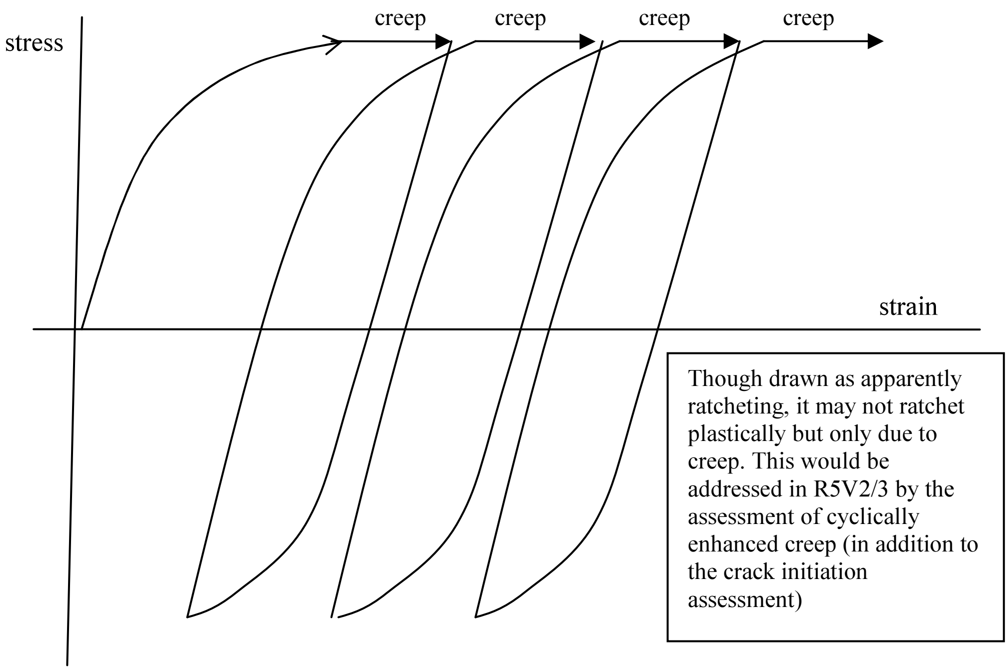


Figure 3b: Hysteresis Loop for Case 2 (large F, ratcheting)

This behaviour might result if the dwell stress were assessed at the rupture reference stress, so there is no relaxation / redistribution. Successive creep damage increments are not the same as forward creep because the creep is perturbed by cycling. This would be assessed in R5V2/3 using ductility exhaustion not creep rupture. Note that the reverse plasticity will tend to reset primary creep in the dwell, enhancing damage compared with true forward creep.



Loading Case 3

A temperature distribution is applied across the cantilever, which would cause an elastic thermal stress of less than $K_S S_y$.

In addition, a load, F, is applied which would cause a rupture reference stress, $\sigma_{SS} = \sigma_{ref}^{R}$, of less than $K_S S_y$.

The total loading causes an elastic stress of greater than $K_S S_y$ but less than $2K_S S_y$. The cycle is,

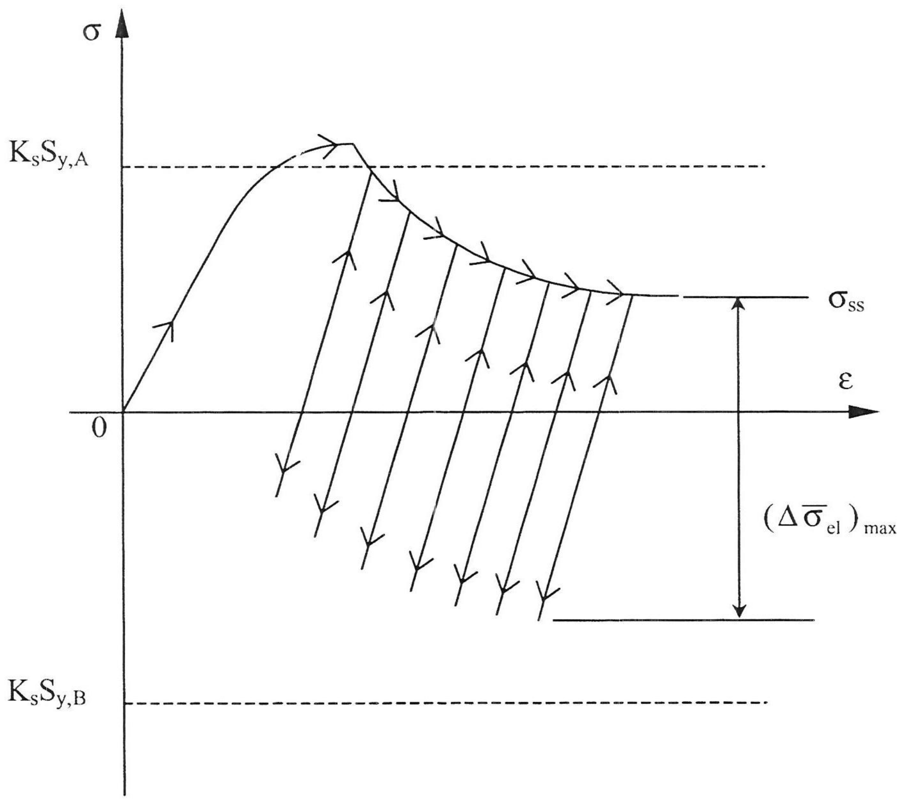
(i) Apply the load, F, and the temperature difference;

(j) Hold the load and the temperature difference constant at high temperature;

(k) Remove both the load and the temperature difference;

(l) Repeat from (a).

Figure 4: Stress-Strain Trajectory for Case 3



Creep stress redistribution or relaxation is undisturbed by reverse yield

Dwell stress relaxes on each successive cycle just as if the load cycles did not occur. But the stress cannot relax below the primary stress level $\sigma_{SS} = \sigma_{ref}^{R}$ due to F.

Fatigue damage will be very slight because the cycles are elastic (no plastic strain after first loading).

After the transient cycling, i.e., once the cycles have achieved a dwell stress of $\sigma_{SS} = \sigma_{ref}^{R}$, the creep is then identical to forward creep and can be assessed as creep rupture, see R5V2/3 §8.3.1. (However, I would add that allowance should be made for the enhanced damage from the earlier cycles, see R5V2/3 §8.3.3).

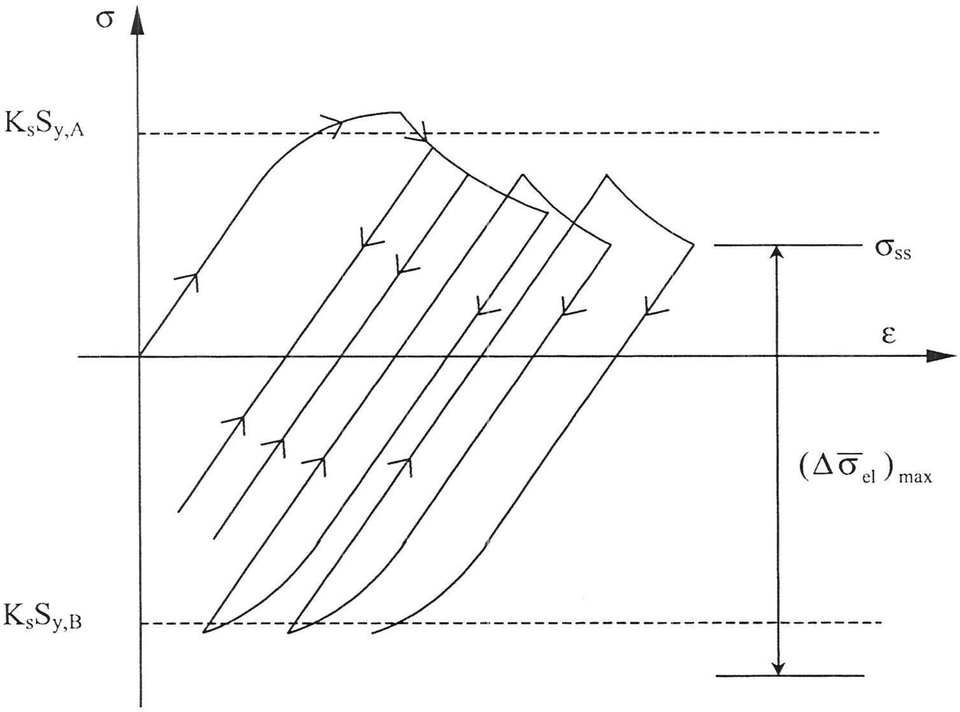
Qu.: Case 4: What happens to Case 3 if we make the thermal load more severe?

The load, F, still corresponds to a rupture reference stress, σ_{SS}, of less than $K_S S_y$.

But the temperature distribution across the cantilever is now made more onerous so that the elastic thermal stress is *greater* than $K_S S_y$, in fact greater than $K_S S_y + \sigma_{SS}$.

However, the total loading still causes an elastic stress of less than $2K_S S_y$. The cycle is now,

Figure 5: Stress-Strain Trajectory for Case 4



Reverse yield repeatedly raises the creep stress in a simple cycle

Initially the stress-strain trajectories are as for Case 3, but eventually on unloading the reverse stress hits the - $K_S S_y$ level. This is because the (elastic) thermal stress, $\sigma_{thermal}$, exceeds $K_S S_y + \sigma_{SS}$, so the stress decrease on unloading exceeds $\sigma_{SS} + K_S S_y$. On re-loading for the next cycle, the dwell stress is thus $-K_S S_y + \sigma_{thermal}$ and this exceeds the primary stress, σ_{SS}, by virtue of $\sigma_{thermal} > K_S S_y + \sigma_{SS}$.

This is an example of how a hysteresis cycle can arise, which re-sets the dwell stress to $\sigma_{Dwell} = -K_S S_y + \sigma_{thermal} > \sigma_{SS}$ on each cycle, despite the fact that the elastic stress range is within the strict shakedown level, i.e., $\Delta\bar{\sigma}^{el} < 2K_S S_y$.

Ductility exhaustion would be used to assess the creep damage per cycle in this case, and primary creep must be reset on each dwell. Physically this is a result of the reversed plasticity.

Qu.: Case 5: Must the creep dwell be at the peak of the cycle?

No.

Suppose a load cycle consists of,

(a) Apply a load whose total elastic stress would exceed $2K_SS_y$;

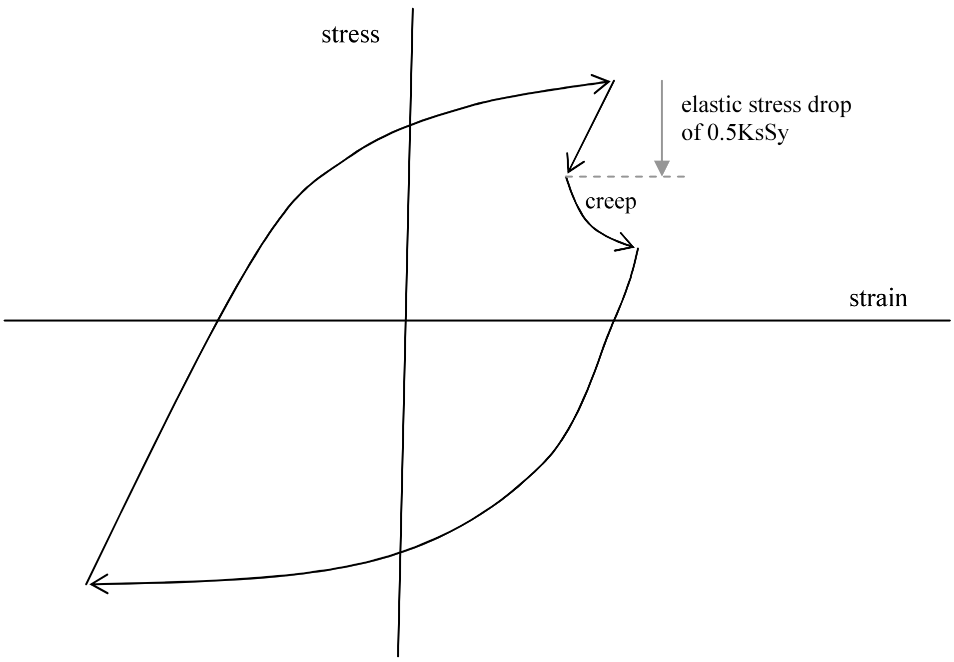
(b) Remove part of the load, correspond to an elastic stress drop of (say) $0.5K_SS_y$;

(c) Dwell at creep temperatures;

(d) Remove all load;

(e) Repeat from (a).

Figure 6: Hysteresis Loop for Case 5



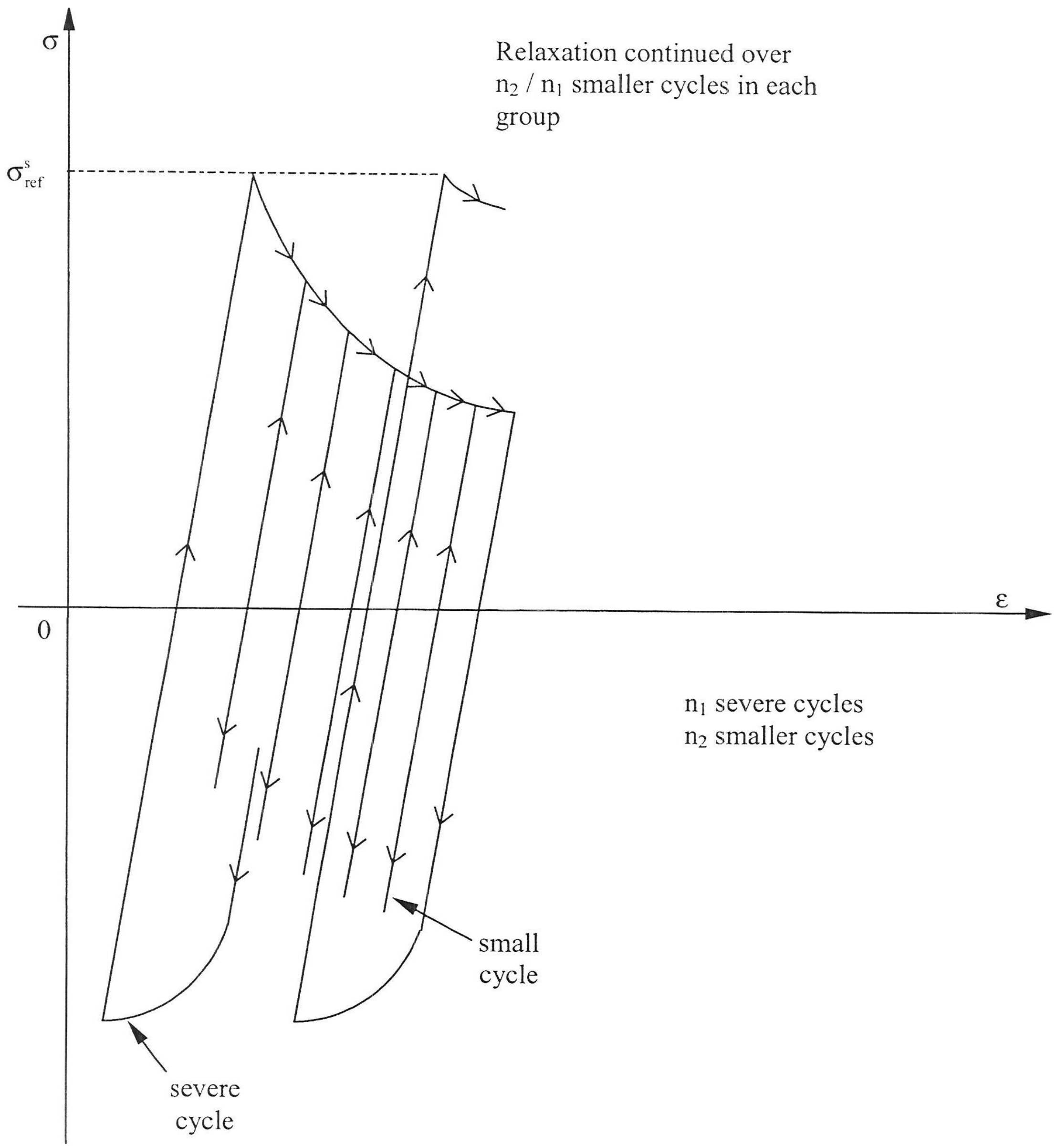
Qu.: What if there are two different types of cycle?

When two different types of cycle occur, in some random sequence, what happens depends upon whether the smaller cycle is elastic or not. The simpler case is when the smaller cycle involves an elastically calculated stress range which is not large enough to cause any plastic straining. Note that this depends upon the stress level due to the larger cycle at that time. See Figure 7 below for an illustration of a secondary cycle small enough to result only in elastic cycling.

In general, suppose the larger cycle results in a stress of $\sigma_{l\arg er}(t)$, at the time, t, that the smaller cycle starts. Suppose that the elastically calculated stress range caused by the smaller cycle is $\Delta\sigma_{smaller}$. For the smaller cycle to be an elastic cycle, it is required that "$\sigma_{l\arg er}(t)-\Delta\sigma_{smaller}$" remain within the 'tram lines' defined by $+(KsS_y)_{top}$ and $-(KsS_y)_{bottom}$. (NB: What is written here as "$\sigma_{l\arg er}(t)-\Delta\sigma_{smaller}$", is to be understood as the Mises stress derived from the difference of the stress components).

Warning: For simplicity of exposition I have been using (KsS_y) as a shorthand for when plasticity occurs. There will be some discrepancy between this criterion and that based on using the mean Ramberg-Osgood curve.

Figure 7: Illustration of Small (Elastic) Cycles Superimposed on Large Cycles



Qu.: How are the smaller cycles assessed?

When the smaller cycles are elastic things are simple:-

The smaller cycles do not perturb the larger cycles, so the creep and fatigue damage due to the larger cycles is calculated as if there were no smaller cycles – using as the effective dwell the period between major cycles (assuming all operating periods have the same temperature).

Similarly the damage due to the smaller cycles can be evaluated without reference to the larger cycles – provided only that their elastic nature has been demonstrated in accord with the above principles.

Since the small cycles are elastic their contribution to the fatigue damage will probably be very small.

If the small and large cycles both involve dwells at the same temperature then it would not matter whether the creep damage were attributed to the large cycles alone, or to the large and small cycles, provided only that the total dwell time was correctly accounted for. However, correct partitioning of the dwell periods into small and large cycle dwells would be important if they involved different temperatures.

Qu.: What if the small cycles are also elastic-plastic?

If both the small and the large cycles involve hysteresis (plastic strain) the life gets more complicated. This will be discussed in the Session 33.

Qu.: How is the dwell stress calculated?

In my opinion the best answer to this is "by hysteresis cycle construction", which will be addressed in the next session.

There are essentially three simple approximations for the dwell stress which do not depend upon hysteresis cycle construction. These are stated in R5 V2/3 Section 7.2.2. They are,

- The first is to set the dwell stress equal to the shakedown reference stress (as defined in session 30). This is only valid if the full definition of σ_{ref}^{s} has been used, including the peak F-stresses across whole structural section.
- The second approximation for the dwell stress is $\Delta\overline{\sigma}_{\text{el,max}} - \text{KsSy}$. This method is pretty useless if the structure is outside strict shakedown, in my view. This is because this dwell stress estimate will generally be excessively large except when the structure is within strict shakedown. If you are in strict shakedown and the hysteresis cycle is like that of Figure 5, then this method does produce the "correct" dwell stress, or at least a sensible estimate.
- The third approximation is to set the dwell stress to KsSy, but with Sy defined by the best estimate yield data rather than the lower bound. However, this is only valid if the result exceeds the dwell stress obtained from hysteresis cycle construction. Since the hysteresis cycle must therefore be constructed anyway, there is therefore no saving of labour. R5V2/3 §7.2.2 does state that you should use the ***larger*** of the dwell stress resulting from hysteresis cycle construction and KsSy based on mean yield. I generally ignore this in favour of the hysteresis cycle construction and I encourage you to commit the same heresy. R5V2/3 refers to this as "closer to a best estimate" start of dwell stress. This is acceptable in the sense that stresses are generally best estimates in structural assessments. Moreover, the assessor should carry out sensitivity studies on the hysteresis cycle construction which will take care of the possibility of optimism.

When outside strict shakedown my advice is to construct the hysteresis cycle according to R5V2/3 Appendix A7 (to be covered in Session 33). However, you must always be mindful of hysteresis cycles like Figure 5 occurring even within strict shakedown.