T73S02 Tutorial Session #19 - J Estimation

Last Update: 22/11/15

J estimation from reference stress; J for secondary stresses; The plasticity correction factor in R6; Difficulties with large secondary stresses, e.g. residual stresses; When do they really contribute to fracture?; Provision in R6 for displacement controlled loading: logic and distinction from secondary loading.

This shading indicates "beyond SQEP requirements"

Qu.: How can we estimate J when plasticity is significant?

In the elastic case we have handbook solutions which can be used to estimate K providing that the loading and the geometry of interest can be approximated sufficiently closely by a standard solution. Supposing this is the case – how can we also estimate J when there are significant plastic strains present? There is a method, called the "reference stress formula" due to Bob Ainsworth.

Qu.: What is the reference stress formula?

From session 17 we know that the R6 failure assessment diagrams (FADs) are really ways of writing the PYFM criterion $J = J_{init}$. Turning this around, the R6 Option 2 material specific FAD can equally be regarded as a J-estimation scheme. The way it works is this: R6 Section I.6.2 gives the Option 2 material specific FAD to be,

$$f(L_r) = \left[\frac{E\varepsilon_{ref}}{L_r\sigma_y} + \frac{L_r^3\sigma_y}{2E\varepsilon_{ref}}\right]^{-\frac{1}{2}} \tag{1}$$

where $L_r\sigma_y = \sigma_{ref}$ is the reference stress and ε_{ref} is the total (elastic+plastic) strain read off the true stress/true strain curve at the reference stress. But we know from session 17 that the FAD abscissa can be interpreted as,

$$f(L_r) = \sqrt{\frac{J_{el}}{J_{el-pl}}} \tag{2}$$

i.e., the square-root of the ratio of the elastic to the elastic-plastic values of J. Consequently, if we have a means of estimating the elastic SIF (and hence the elastic J), and we know what the stress-strain curve is, then we can use (1) and (2) to derive the value of the elastic-plastic J. Explicitly this gives,

$$J_{el-pl} = \left[\frac{E\varepsilon_{ref}}{\sigma_{ref}} + \frac{\sigma_{ref}^3}{2E\sigma_y^2\varepsilon_{ref}}\right] J_{el} \tag{3}$$

Qu.: How were these expressions arrived at? A History Lesson…

In the 1970s the American Electrical Power Research Institute (EPRI) commissioned work to evaluate J for some idealised geometries and for power law hardening. Explicit expressions for J in terms of stress were derived and published. Bob Ainsworth noticed that by substituting $\varepsilon/\varepsilon_y$ for $(\sigma/\sigma_y)^n$ where-ever the latter appeared, the expressions could effectively be extrapolated to apply for an arbitrary stress-strain curve. Moreover, the geometry dependent parts of the EPRI expressions could be absorbed into the definition of the stress, i.e., by substituting reference stress for applied stress the expressions lost their explicit geometry dependence. The result was (3), an expression for J which applies, at least to a good approximation, for any geometry or material. From this the R6 Option 2 expression, (1) was derived (not the other way around!). Finally, by plugging the stress-strain curve for a whole range of relevant materials into the Option 2 expression, (1), a bundle of FADs was drawn. A curve towards the lower bound of this spread of individual FADs was then drawn, so as to be conservative for almost all materials. This is how the Option 1 (generic) R6 FAD was derived. So, it was all the opposite way around from the way it is now presented.

Qu.: What does the reference stress formula for J mean physically?

Consider firstly what (3) gives us when the reference stress is less than the stress at first deviation from linearity, σ_0. In this case the first term is just 1 and we get,

$$J_{el-pl} = \left[1 + \frac{L_r^2}{2}\right] J_{el} = \left[1 + \frac{1}{2}\left(\frac{\sigma_{ref}}{\sigma_y}\right)^2\right] J_{el} \tag{4}$$

This correction to the LEFM value of J is known as the small-scale yielding correction – because it applies when plasticity is confined to near the crack tip (so that the gross ligament is still elastic, $\sigma_{ref} < \sigma_0 < \sigma_y$). The physical meaning of the correction term is elucidated by recalling that the Mode I stress ahead of the crack is,

$$\sigma_I = \frac{K}{\sqrt{2\pi r}} \tag{5}$$

Hence the distance ahead of the crack over which the Mode I stress exceeds yield is,

$$r_y = \frac{1}{2\pi}\left(\frac{K}{\sigma_y}\right)^2 \approx \frac{a}{2}\left(\frac{\sigma}{\sigma_y}\right)^2 \tag{6}$$

So that,

$$J_{el-pl} \approx \left[1 + \frac{r_y}{a}\right] J_{el} \qquad \text{when } r_y << a \tag{7}$$

In (6) and (7) I have ignored the compliance factor. Also, in (7) I have fudged the distinction between the applied stress and the reference stress. So these expressions are only approximate. The point is only to show, via (7), that the correction to the LEFM J is small so long as the yield zone (which is roughly of the same order as r_y) is small compared to the crack length, *a*.

Qu.: What about when the reference stress approaches or exceeds the yield stress?

In this case the first term in (3) becomes larger than unity. In fact once the reference stress passes the yield stress, the reference strain increases rapidly and so does this first term. In contrast, the second term reduces. The first term dominates when $\sigma_{ref} > \sigma_y$ and we can approximate,

For $\sigma_{ref} > \sigma_y$

$$J_{el-pl} \approx \left[\frac{E\varepsilon_{ref}}{\sigma_{ref}}\right] J_{el} = \left[\frac{\varepsilon_{ref}}{\varepsilon_{ref}^{el}}\right] J_{el} \tag{8}$$

Defining the effective elastic-plastic SIF as $K_{eff} = \sqrt{EJ/(1-\nu^2)}$ we can write this as, For $\sigma_{ref} > \sigma_y$

$$K_{eff} \approx \left[\frac{E\varepsilon_{ref}}{\sigma_{ref}}\right]^{\frac{1}{2}} K = \left[\frac{\varepsilon_{ref}}{\varepsilon_{ref}^{el}}\right]^{\frac{1}{2}} K \qquad (9)$$

These expressions show that *J*, and K_{eff}, increase very rapidly once the stress exceeds the yield stress – and that this is a consequence of the rapidly increasing plastic strains. It is the plastic strain which is responsible for the large *J*.

Qu.: Does the reference stress formula, (3), apply for any sort of loading?

No.

It applies only for primary loading.

This is clear because only primary loads contribute to σ_{ref}, and hence to ε_{ref}. So, if we consider a secondary load alone, Equ.(3) would imply that *J* was always equal to its LEFM value for secondary stresses. In fact, in many cases, this might be a reasonable approximation. But, in general, secondary plastic strains will affect the value of *J*.

However, unlike the case of purely primary stressing, for a purely secondary stress the resulting plastic strain might either increase or decrease the value of *J* compared to its elastic value.

Qu.: Can we devise a J-estimation scheme that works for secondary stresses too?

Yes. But it's a lot more involved. Before we get to that, let's see how R6 handles the matter. And before we get to that, what would we *expect* the elastic-plastic J under secondary loading to be?

Qu.: Elastic-plastic J for secondary load?

There is no simple answer, but the problem can be viewed like this…

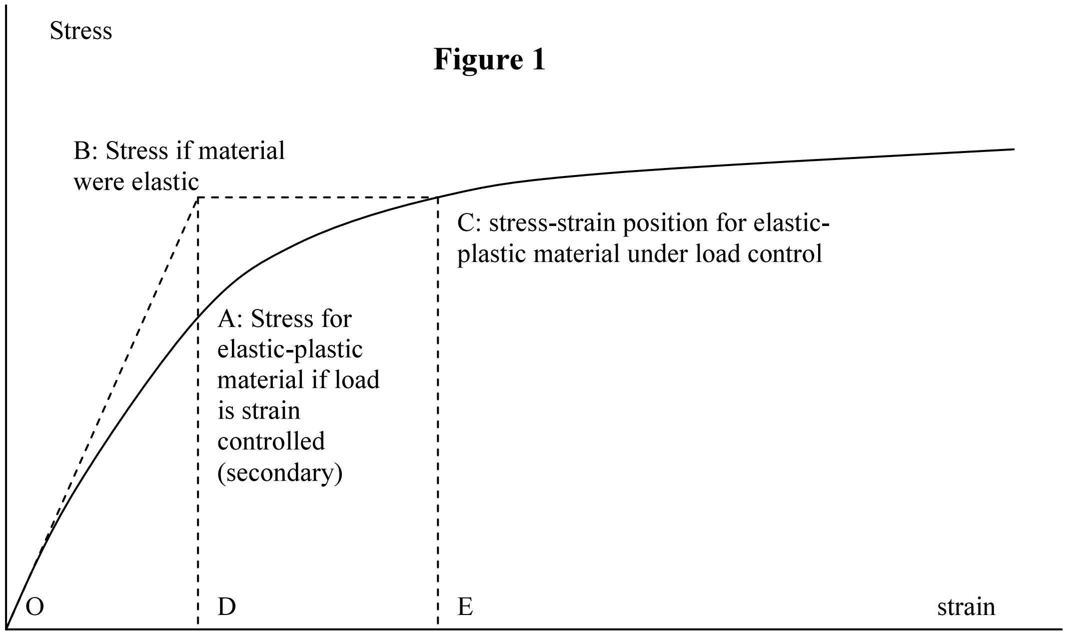


Figure 1

Recall from the homework of session 15 that, as a rough rule-of-thumb, the elastic-plastic J is proportional to the area under the load-displacement curve – which might be roughly proportional to the area under the above stress-strain curve. So we would expect the relative magnitudes of the J's for (a)elastic-plastic strain control, (b)elastic material, and, (c)elastic-plastic load control, to be in the order of the corresponding areas,

$$OAD < OBD < OACE \qquad (10)$$

i.e., $$J^{el-pl}_{straincontrol} < J^{el} < J^{el-pl}_{loadcontrol} \qquad (11)$$

Well, this *might* be right. The elastic-plastic value of J under strain control *can* be less than the elastic value – for the above reason. But not always – and not even typically.

Qu.: Why?

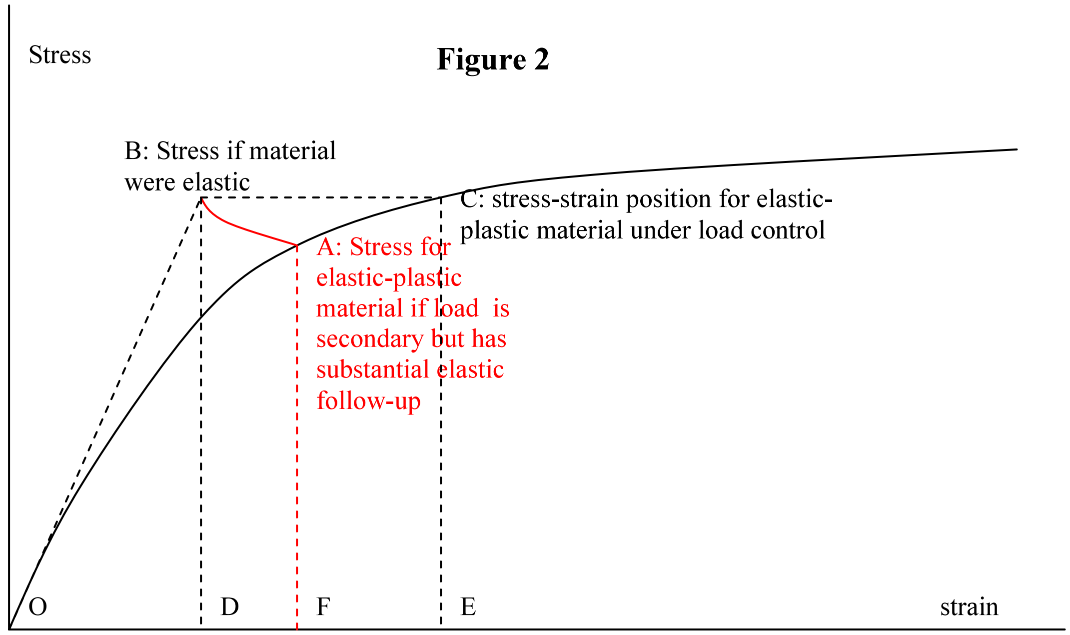
The reason is that the above graph is really just about the applied stress or strain, remote from the crack. But J is a measure of the crack tip fields. So what if we consider a point near the crack tip? The difference is that, although for secondary loading the remotely applied strain (or displacement) may be fixed, we cannot assume the strain at an arbitrary point is fixed: quite the opposite. According to Neuber's rule-of-thumb, near a notch feature the strain would have to increase so that the product of stress and strain remained the same as the stress relaxes plastically,

Neuber: $$\sigma_{el-pl}\varepsilon_{el-pl} = \sigma_{el}\varepsilon_{el} \qquad (12)$$

Crudely, from the integral definition, J depends upon terms like stress times strain – so according to Neuber's rule we might expect the elastic-plastic value of J under strain control to be about the same as the elastic value. This is getting closer to the truth now – but in fact, rather inconveniently, $J^{el-pl}_{straincontrol}$ can be either smaller than, or equal to, or greater than the elastic J.

Qu.: How can $J^{el-pl}_{straincontrol} > J^{el}$?

Neuber's rule implicitly assumes a certain degree of elastic follow-up. But the elastic follow-up really depends upon the details of the loading and geometry. For example, a large displacement applied over a commensurately long specimen is hardly a secondary load at all. It is closer to primary, and this is manifest as a large elastic follow-up factor, Z. The behaviour at a point near the crack tip then looks like,



$J^{el-pl}_{straincontrol}$ is now related more to OAF, and hence can exceed J^{el} which is related to OBD.

So, we get $J^{el-pl}_{straincontrol} < J^{el}$ if Figure 1 is most representative.

Or, we get $J^{el-pl}_{straincontrol} > J^{el}$ if Figure 2 is most representative.

Or, we get $J^{el-pl}_{straincontrol} \approx J^{el}$ if Neuber is most representative.

However, the one thing that's always true is $J^{el-pl}_{straincontrol} \leq J^{el-pl}_{loadcontrol}$.

Of course, the above "explanations" are very heuristic and are only meant to illustrate in a simple appealing manner a more complicated reality.

Qu.: OK, so how does R6 handle secondary loads?

Firstly recall how R6 handles the effect of plasticity due to *primary* loading. This is done through the Lr parameter. The y-axis of the FAD involves just the elastic SIF (divided by the toughness), Kr. So primary plasticity is addressed via the Lr parameter, together with the fact that a smaller value of Kr is permitted as Lr increases.

We cannot address the effect of secondary load induced plasticity in this way, because, by definition, only primary loads contribute to Lr. And we do not want to have to change the FAD itself according to the applied secondary loading (though this would be a possible approach). Instead, R6 opts for a correction to Kr to account for secondary plasticity. This may be an additive term (ρ) or a factor (V), thus, Either $$Kr \to Kr^P + Kr^S + \rho \tag{13a}$$

Or $$Kr \to Kr^P + VKr^S \tag{13b}$$

where the superscripts ^{P} and ^{S} refer to the primary and secondary loads respectively.

> NB: In the 2015 revision of R6 the additive method using ρ has been withdrawn. I have retained it in these notes for the present but it is no longer within R6.

The two methods are broadly equivalent and which you use is a matter of taste. Note that it is necessary in R6 to categorize the loading as primary or secondary, and will in general be a mixture of the two.

Qu.: What are the magnitudes of ρ and V?

R6 gives two methods for calculating the values of ρ and V:-

[1] The simpler method is also more conservative. The possible ranges of values are $0 \le \rho \le 0.25$ and $1 \le V \le 1.39$. (I've worked out the upper limit on V myself – I invite you to check it). This means that the possibility represented by Figure 1, that the elastic-plastic J is actually *less than* the elastic value is not catered for in this conservative estimate. In this method, secondary plasticity can only make things worse.

[2] A slightly more involved method (but not very difficult) is also given which does allow for the possibility that $\rho < 0$, or equivalently, $V < 1$. These cases realise Figure 1, i.e., that $J^{el-pl}_{straincontrol} < J^{el}$. It is important to remember this – it can be a very helpful alleviation when assessing large secondary loads.

Qu.: How were the R6 formulations for ρ and V worked out?

Initially there were some rather approximate analytical estimates. But more recently the R6 formulations have been underpinned by a range of finite element calculations (and, I suspect, some test work as validation – but consult R6 to see).

Qu.: Residual plus Other Secondary Stresses Greater Than Yield

An important special case in R6 occurs when there is an applied secondary load as well as a residual stress field (which is also secondary), and their combination exceeds yield. In this case R6 permits the plastic strain associated with the residual stress part to be ignored. See R6 II.6.8 for details. This is rather helpful since the strain associated with welding residual stresses is virtually indeterminate – or, at least, very complicated and usually very large.

Qu.: What is the interaction between primary plasticity and secondary plasticity?

In the R6 formulations you find that $\rho = 0$ and $V = 1$ (i.e., the secondary plasticity correction vanishes) when Lr is sufficiently large (typically for Lr > 1.05). Physically this means that once general yielding occurs due to primary loading alone, the secondary plasticity effects get "washed out" by the primary strains.

Qu.: Can residual stresses really cause fracture?

Yes.

For example, when some of the early Magnox reactor pressure vessels were being fabricated, some 'strakes' spontaneously fractured after welding – with no primary loading present (there was probably some thermal stress too, though).

Qu.: How do R6 and R5 compare regarding how they address elastic follow-up?

The R6 procedures for accounting for the effects of plasticity due to secondary stresses (ρ and V) are, in effect, a *partial* means of incorporating elastic follow-up – though it is not usually expressed in this way. However, the degree of follow-up which is implicit is modest, the ρ and V procedures being essentially for 'pure' secondary loads. However, R6 also includes specific provision in Chapter III.14 for displacement controlled loads, which will generally include elastic follow-up effects. Although this procedure does not refer explicitly to elastic follow-up, it implicitly covers the phenomenon for the case of pure displacement controlled loads.

In contrast, R5 includes elastic follow-up explicitly through the use of a Z factor in the procedures.

Qu.: But aren't displacement controlled loads just secondary loads?

Not necessarily.

A secondary load has little elastic follow-up. In contrast, a displacement controlled load can have any degree of elastic follow-up (Z may be anywhere between 1 and infinity). A displacement controlled load with Z = 1 is secondary. But a displacement applied over a very long gauge length can have such a large Z that it approximates to a primary load. R6 Chapter III.14 addresses the general case of displacement controlled loading – but only if there are no other types of load present as well.

Qu.: Is there a J-estimation method for general loading?

Yes. But it requires a bit of work in the form of numerical differentiation and integration. The key observation is that, so long as we know what the current loads are, we can use equation (3) to find the elastic-plastic J from the elastic SIF solution. This is because, given the current loads, we can evaluate the reference stress and hence the reference strain. This assumes that we also have an appropriate collapse solution – but this would be required in any R6 assessment anyway. The reason that this works is that structure does not know about our categorisation of loads into primary and secondary loads. The structure just responds to the current load acting on it. The difficulty with secondary (or displacement controlled) loads is that we do not initially know the *loads* (i.e., the forces and moments), only the applied displacements or strains. If we had a means of calculating the loads corresponding to the applied strains or displacements then the problem would be solved. But we need these loads for the cracked body, and the crack itself – as well as the plastic strains – will reduce the loads. The stiffness of the structure reduces due to plasticity and due to the crack.

But we already know how to find the reduction in the stiffness, and hence in load, of a structure due to a crack (see Homework 15). The secret is that J is the energy release rate, and energy is related to load. So we can devise a means of estimating the load as long as we know how to find J at a given load – which we do. The general method has been presented in my paper Engrg.Fract.Mech. **29** (1988) 683-696. The result is as follows. Suppose a displacement *D* and a rotation θ are applied to the structure, and these result in a force resultant *F* and bending moment *M*. For simplicity, suppose we have a 2D body of thickness *t*. The displacement/rotation are related to the force/moment by,

$$D = D_0 + t\int \frac{\partial J}{\partial F} da \qquad \theta = \theta_0 + t\int \frac{\partial J}{\partial M} da \qquad (14)$$

The integrals over the crack length in (14) extend from 0 to the crack length of interest. The value of *J* in the integrands is found for the assumed loads, *F* and *M*, from Equ.(3). This allows the derivatives with respect to *F* or *M* to be found (it is simplest to do so numerically in practice). The terms D_0, θ_0 are the displacement and rotation for the uncracked body corresponding to the assumed loads, *F* and *M*. In general these will be elastic-plastic values, but it is simplest and conservative to use the elastic values. Hence they can be found from the loads via the elastic stiffness of the uncracked body.

As written, (14) will give the displacement and the rotation of the cracked body for given loads *F* and *M*. We want to find the loads given the displacement and rotation. Various algorithms are possible to get around this. One is simply to iterate values for *F* and *M* until the RHSs of (14) equal the prescribed LHSs. In practice the method must be implemented by some computer code – so it is a lot more complicated than merely using Equ.(3) for primary loads. On the other hand, it provides an alternative to full finite element analysis – and sometimes FEA has problems with secondary stresses, particularly in the evaluation of J contour integrals.

Qu.: Why is an FEA evaluation of J sometimes problematical for secondary loads?

The problem arises in conventional formulations of the J integral when there are plastic strains present before the crack is introduced. Recall that J is only path independent, and hence meaningful, for non-linear elasticity really. When the irreversible effects of plasticity enter the problem, all bets are off. So, for example, if a welding residual stress field is modelled (which will involve plastic strains by definition) and then a crack is introduced – expect trouble when evaluating J. There are specialist codes which are supposed to do so – but that would be too long a digression.