T73S02 Tutorial Session #16: J Derived from Noether's Theorem

Not required for SQEP

The derivation of J as an example of Noether's theorem; the physical meaning of its contour independence and the physical reason why this can break down in incremental plasticity. J_2 and J_3, what does ABAQUS VCE mean?

Last Update: 27/9/2015

Qu.: Haven't we already derived J previously?

Not really. In session 15 we just wrote down the integral definition of J and then proved that it equals the energy release rate (for proportional loading). That's just what Rice did in 1968. But where did this rather obscure integral come from? It was derived originally (and I do mean *derived*) by John Eshelby, an English physicist, in 1951. Rice referenced Eshelby in his later, 1968, paper – but its Rice's paper which has become the one everyone quotes.

Qu.: What is the history of J?

- J.D.Eshelby, "The Force on an Elastic Singularity", *Phil.Trans.Roy.Soc.London* A**244**, 87-111 (1951).

Although Eshelby's interests were in lattice defects, he is explicit about the equal relevance of his investigations to continuum theory as to dislocations. He used the Noether approach which we will discuss here, and referred to the energy-momentum tensor as the "Maxwell tensor of elasticity", by analogy with electromagnetic theory. Of course, the "force" on the singularity is just the energy release rate (per unit thickness). So Eshelby set out to find an expression for the energy release rate – and found it via Noether. The discussions in the paper give every impression that this is the original source of the J integral. The 1951 paper, however, refers only to elastic behaviour. But in a subsequent paper in 1956, Eshelby made it clear that the same result followed, "for finite strain and an arbitrary stress-strain relation", by the exact same Noether method.

- J.D.Eshelby, "The Continuum Theory of Lattice Defects", Solid State Physics 3, 79-144 (1956), Academic Press, Elsevier.

Consolidating the early works, in 1959 Eshelby also proved the equality of the J integral and the energy release rate for an arbitrary stress-strain curve using the method of session 15 (which was also his invention).

- J.D.Eshelby, "Scope and Limitations of the Continuum Approach", in *Internal Stresses and Fatigue in Metals*, eds. G.N.Rassweiler and W.I.Grube, pp. 41-58, 1959, Elsevier, Amsterdam.

Qu.: So, will we just address "where J comes from" in the mathematical sense?

No. The real motivation of this session is that deriving J from Noether's Theorem tells us what the path independence of J means physically. It also puts J into context with the rest of continuum physics.

Qu.: What is Noether's Theorem?

I will spare you the proof – the derivation of J, below, will suffice. But the theorem is this: -

Noether's Theorem: If a system has a continuous symmetry, then there is a related quantity which is conserved

Qu.: What examples are there?

Energy

We all know that energy is conserved – but why? The amazing thing is that the conservation of energy can be proved – it is a mathematical theorem. It's an example of Noether's theorem. The symmetry in question is translation in time. What is meant by a system being invariant under a translation in time? It simply means that the laws which the system obeys are the same now as they were yesterday, and will be again tomorrow. This symmetry (invariance) implies a conserved quantity exists. This conserved quantity is energy.

The fact that the laws of physics are the same today as they were yesterday, and will be again tomorrow, implies that energy is conserved

Momentum

The conservation of momentum also follows from Noether's theorem. The symmetry in question is invariance under translations in space. Providing the laws obeyed by the system are the same in Gloucester as they are in Manchester, or anywhere else, then a conserved quantity exists. This is momentum. The conservation of momentum is due to the homogeneity of space.

The fact that the laws of physics are the same here as everywhere else in the universe implies that momentum is conserved

Angular Momentum

The conservation of angular momentum also follows from Noether's theorem. The symmetry this time is invariance under rotations. Providing the laws obeyed by the system are the same however we orient the system in space, then a conserved quantity exists. This is angular momentum. The conservation of angular momentum is due to the isotropy of space.

The fact that the laws of physics make no distinction between one direction and another implies that angular momentum is conserved

Electric Charge

The conservation of electric charge also follows from Noether's theorem. In this case the symmetry is not a spacetime symmetry. It involves the invariance of the theory of electromagnetism under certain algebraic transformations (gauge transformations). More complicated (non-abelian) gauge symmetries are responsible for the conservation of other quantum numbers in particle physics (e.g., weak isospin, i.e., 'weak charge', and quark/gluon colour charge).

Qu. Does Noether's Theorem apply to any type of physical theory at all?

Not quite. To be precise, Noether's theorem assumes that the theory in question is expressible in a certain form, known as the Lagrangian formulation. All fundamental physics can be expressed in this form, providing that we are dealing with a conservative (non-dissipative) system. This really means that we include sufficient detail so that we can keep track of all the energy.

Noether's Theorem ranks as the most stunning theorem in mathematical physics. Why is it not more widely known?

Qu.: What does Noether's Theorem tell us about J?

First you have to ask – what is the symmetry?

The symmetry is the (assumed) homogeneity of the material.

If we have a homogeneous material, then the equations which describe its physical behaviour must be invariant under translations in space. We could be talking of any physical properties – such as electromagnetic properties, for example. But we are interested in stress and strain. If we confine attention to statics, Noether's Theorem tells us…

The homogeneity of the material implies the path independence of J

Qu.: What does this tells us about the limitation to the path independence of J?

It explains why path independence breaks down after yielding if we allow any unloading. The irreversibility means that the previously-yielded material no longer behaves in the same way as the unyielded material. In effect it has become a different material. The material is therefore no longer homogeneous, and the reason for the path independence has gone.

Qu.: But I thought that something was going to be conserved?

Path independence is actually closely linked to conservation.

You only get a conserved quantity, in the sense of no variation in time, if time comes into your theory – obviously! Conservation laws only have relevance in dynamics. If we consider the dynamic version of stress analysis, then Noether's Theorem gives us the conservation of energy and momentum of stress waves propagating in the material.

Path independence is what a conservation law reduces to in the static limit.

Qu.: How does conservation give us path independence?

To understand this, you have to appreciate how to formulate conservation when the stuff in question is distributed all over the place and can move around. In the Appendix this is explained, and it is shown how conservation reduces to the vanishing of a divergence in the static case. A vanishing divergence is what gives rise to a path independent integral.

Qu.: Do we have to use a Lagrangian formulation?

No.

In the general (dynamic) case, the demonstration of Noether's Theorem would require the theory to be formulated in terms of Lagrangian dynamics. However, it is sufficient for our purposes to consider static non-linear elasticity. Because non-linear elasticity is conservative (i.e. non-dissipative) there exists a strain energy density, W, which can be written as a function of the strains. The work done in increasing the strain by $\delta\varepsilon_{ij}$ over a region of volume δV where the stress is σ_{ij} is $\sigma_{ij}\delta\varepsilon_{ij}\delta V$ (where repeated indices are summed unless otherwise indicated). This must equate to the increase in the energy density times the volume, $\delta W \delta V$, for a conservative system. Hence,

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{1}$$

Since the strains are given in terms of the displacement gradients by,

$$\varepsilon_{ij} = \frac{1}{2}u_{(i,j)} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

we can equally regard W as a function of the $u_{i,j}$. Hence,

$$\sigma_{ij} = \frac{\partial W}{\partial u_{i,j}} \tag{3}$$

But equilibrium gives us: $\sigma_{ij,j} = -b_i$ (4)

So, $$\partial_j\left(\frac{\partial W}{\partial u_{i,j}}\right) = -b_i \tag{5}$$

But the force per unit volume is given by (minus) the gradient of the potential energy, i.e., $b_i = -\frac{\partial W}{\partial u_i}$. So (5) becomes,

$$\partial_j\left(\frac{\partial W}{\partial u_{i,j}}\right) = \frac{\partial W}{\partial u_i} \tag{6}$$

Equ.(6) represents three equations, for the three possible values for i, noting that the j is summed.

(If equs.(6) had been derived via a Lagrangian approach, they would be called the Euler-Lagrange equations of the system).

Qu.: Where does homogeneity come in?

Patience! We're now all set up for the magical appearance of J.

For a body under load, the energy density, W, will generally vary from point to point. In what sense, then, is the material homogeneous?

What is meant by homogeneity is that the underlying *theory* is not biased to one point compared with another. Any particular *solution* to the equations, i.e., for a given set of boundary conditions, will generally be inhomogeneous in the sense that the stresses

and strains will vary from place to place. The distinction between the homogeneity of the theory, as contrasted with the spatially varying specific solution, is manifest in the distinction between *partial* derivatives and *total* derivatives. The homogeneity of the material requires that the energy density, W, has no *explicit* dependence upon position. Hence its partial derivatives will vanish, i.e.,

$$\partial_i W \equiv \frac{\partial W}{\partial x_i} \equiv 0 \tag{7}$$

On the other hand, for a body under some arbitrary loading, the magnitude of the energy density will obviously vary from place to place in general. The reason is that W depends upon the strains, and the strains will vary from place to place. This is written in terms of the total derivative as,

$$\frac{dW}{dx_i} \neq 0 \text{ in general} \tag{8}$$

The total derivative can be written using the chain rule as,

$$\frac{dW}{dx_i} = \frac{\partial W}{\partial x_i} + \frac{\partial W}{\partial u_j} \cdot \frac{\partial u_j}{\partial x_i} + \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_{j,k}}{\partial x_i} \tag{9}$$

But using (6), $\frac{\partial W}{\partial u_j} = \partial_k \left(\frac{\partial W}{\partial u_{j,k}} \right)$, this becomes,

$$\frac{dW}{dx_i} = \frac{\partial W}{\partial x_i} + \partial_k \left(\frac{\partial W}{\partial u_{j,k}} \right) \cdot \frac{\partial u_j}{\partial x_i} + \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_{j,k}}{\partial x_i} = \frac{\partial W}{\partial x_i} + \partial_k \left[\frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial x_i} \right] \tag{10}$$

Re-arranging and introducing the Kronecker delta, δ_{ik}, we have,

$$\frac{d}{dx_k} \left[W \delta_{ik} - \frac{\partial W}{\partial u_{j,k}} \cdot u_{j,i} \right] = \frac{\partial W}{\partial x_i} \tag{11}$$

But homogeneity means that this is zero, by Equ.(7). Defining the tensor ξ_{ik} by,

$$\xi_{ik} = W \delta_{ik} - \frac{\partial W}{\partial u_{j,k}} \cdot u_{j,i} \tag{12}$$

the homogeneity of the material is thus expressed by the fact that this tensor has zero divergence, i.e.,

$$\xi_{ik,k} = 0 \tag{13}$$

Homogeneity implies there is a tensor, ξ_{ik}, with zero divergence, $\xi_{ik,k} = 0$

The alert reader will have spotted in Equ.(12) the origin of the arcane-seeming integrand which defines J,

$$J_1 = \int_{\Gamma_1} \left[W dy - \overline{T} \cdot \frac{\partial \overline{u}}{\partial x} ds \right]$$

Qu.: That's all very well, but I don't see any path-independent contour integrals?

Ah, but we now have a quantity with vanishing divergence, and that's almost the same thing.

It is?

Yes. Watch. The tensor ξ_{ij} clearly has units of energy density. So, we first define a vector quantity by integrating the divergence of ξ_{ij} over some arbitrary volume, *R*, within the body,

$$\Im_i = \int_R \xi_{ij,j} dV = 0 \tag{14}$$

So $\Im_i$ has units of energy/length. But the volume integral of a divergence is just the surface integral over the boundary of the region, δR. Hence we can write,

$$\Im_i = \oiint_{\delta R} \xi_{ij} dS_j = 0 \tag{15}$$

where the notation emphasises that this is an integral over (any) closed surface lying completely within the material.

Qu.: Well, this is all very merry, but isn't a quantity which is identically zero rather useless? And in any case, shouldn't we have a crack somewhere?

We shall now introduce a crack, and also restrict attention to 2-dimensional cases to begin with. The vanishing of $\Im$ will be seen to be the origin of the path independence of *J*.

As is conventional, place the crack along the negative x-axis, with the tip at the origin, so that the y-stress is the Mode I opening stress. Because the problem is now 2D, the closed surface, δR, is prismatic. It consists of some closed boundary Γ in the x,y plane (actually a prismatic surface), together with the two "ends caps" at some pair of constant values of z.

Consider i = 1 in (12). Substitute this into (15) and use (3), giving,

$$\Im_1 = \oiint_{\delta R} \left[W\delta_{1k} - \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial x} \right] dS_k = \oiint_{\delta R} \left[W\delta_{1k} - \sigma_{jk} \cdot \frac{\partial u_j}{\partial x} \right] dS_k = 0 \tag{16}$$

Consider a 2D problem: consider firstly the end caps (at some *z* = constant)...

The element of surface area *dS* on the ends caps is parallel to the *z*-axis, and hence has no x-component. Hence the first term in the integrand of (16) is zero. The second term involves the traction $\sigma_{jk} dS_k$ acting over the element of area dS. But for a 2D plane stress problem the traction on the ends caps is zero. Alternatively, for a 2D plane strain, or engineering plane strain, problem, the z-displacement is uniform, so that $\frac{\partial u_z}{\partial x} = 0$. The third possibility is anti-plane strain. In this case the tractions $\sigma_{xz} dS_z$ and $\sigma_{yz} dS_z$ will be non-zero, but will be equal and opposite on the two end caps, giving zero net contribution. **In any 2D case, therefore, the end caps contribute nothing to $\Im_1$.**

Consider an element (dx,dy) of the boundary Γ in the x,y plane….

The corresponding element of surface area is $dS = t(dy,-dx)$, where t is the thickness of the body in the z-direction. Hence, the first term in the integrand of (16) involves $dS_1 = tdy$. The second term can be written in terms of the unit vector normal to the surface, $\hat{n}$, where $d\bar{S} = \hat{n}tds$. Hence we get,

$$\Im_1 = \oint_\Gamma \left[tWdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} tds \right] = 0 \qquad (17)$$

where ds is an element of length along the boundary Γ, and dy is its y-component. The thickness, t, factors out and we finally define the conventional J integral by,

$$J \equiv J_1 = \frac{\Im_1}{t} \qquad (18)$$

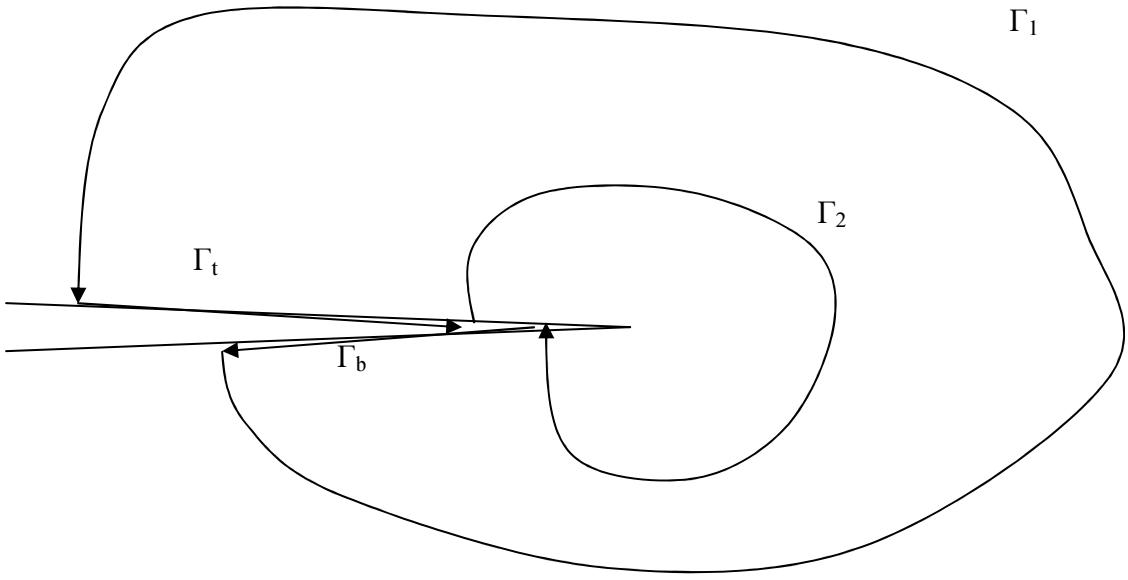
So J has units of energy/area, as required, and is given by the usual expression,

$$J_1 = \oint_\Gamma \left[Wdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds \right] = 0 \qquad (19)$$

But, this is still identically zero, I hear you cry!

A mere detail. This is due to the choice of boundary, Γ, namely a closed boundary. Consider now a closed boundary constructed in the following special manner:-

- Γ_1 is a boundary which begins on the lower negative x-axis, and ends on the upper negative x-axis;
- Γ_2 is a boundary which begins on the upper negative x-axis, and ends on the lower negative x-axis, and lies entirely within Γ_1;
- Γ_t runs along the upper negative x-axis and joins Γ_1 to Γ_2;
- Γ_b runs along the lower negative x-axis and joins Γ_2 to Γ_1.



The final step consists of noting that *dy* is zero along Γ_t and Γ_b, and that the traction $\sigma_{jk}\hat{n}_k$ is also zero on these free surfaces of the crack. Hence Γ_t and Γ_b contribute nothing to the integral ***provided that the crack is straight and lies along the x-axis***.

The fact that the closed boundary integral is zero therefore means that the integrals along Γ_1 and Γ_2 are equal and opposite. But note that the contour Γ_2 is defined as clockwise, whereas Γ_1 is anti-clockwise. If we redefined Γ_2 to run anti-clockwise as well, then the integrals over both would be the same. Hence, we finally have the true, contour independent definition of the J integral,

$$J_1 = \int_{\Gamma_1}\left[Wdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds\right] = \int_{-\Gamma_2}\left[Wdy - \sigma_{jk}\hat{n}_k \cdot \frac{\partial u_j}{\partial x} ds\right] \qquad (20)$$

The traction vector is $T_j = \sigma_{jk}\hat{n}_k$ so an alternative notation is,

$$J_1 = \int_{\Gamma_1}\left[Wdy - \bar{T} \cdot \frac{\partial \bar{u}}{\partial x} ds\right] \qquad (21)$$

Note that the reason why the integral is not zero is because it is not really a closed contour. The presence of the crack prevents the contour being closed, since one end of the contour lies on the top crack face and the other end lies on the bottom crack face. If we began and ended the contour on the same crack face, the integral would be zero. The integral is non-zero by virtue of surrounding the crack tip.

Equ.(20) expresses the path independence, since the value of the integral on contours Γ_1 and Γ_2 are the same, if they are integrated in the same sense (anti-clockwise). But we can fix the contour Γ_2 to be a vanishingly small circular contour centred on the crack tip whilst taking Γ_1 to be any larger contour starting and finishing on the top and bottom crack faces respectively. It follows that the contour integral around any such Γ_1 must be the same.

Note that the path independence results from the fact that the integral over a truly closed boundary, formed by combining the contour Γ_1 with other contours as in the diagram, is zero. And the integral over a truly closed boundary is zero because of the homogeneity of the material.

Qu. But all this followed from setting i = 1 in Equ.(12). What about i = 2 or 3?

This would give us the J_2 and J_3 Integrals. The J-integral is actually a vector!

The $\bar{J}$ -integral is actually a vector!

Consider i = 2 in (12). Substitute this in (15) and use (3) to give,

$$\Im_2 = \oiint_{\delta R}\left[W\delta_{2k} - \frac{\partial W}{\partial u_{j,k}} \cdot \frac{\partial u_j}{\partial y}\right] dS_k = \oiint_{\delta R}\left[W\delta_{2k} - \sigma_{jk} \cdot \frac{\partial u_j}{\partial y}\right] dS_k = 0 \qquad (22)$$

For a 2D problem and a prismatic surface, the ends caps again give no contribution to the integral for the same reasons as before. Since dS = t(dy,-dx) the first term in the integrand now involves –tdx. The second term involves the traction over the surface element, as before. It is clear that, in analogy to Equ.(21), we end up with,

$$J_2 = -\int_{\Gamma_1+\Gamma_t+\Gamma_b}\left[Wdx + \overline{T}\cdot\frac{\partial\overline{u}}{\partial y}ds\right] \tag{23a}$$

Note, however, that (23a) has retained the contribution to the integral along the top and bottom crack faces. The second term in the integrand is zero along these paths, because the crack faces are assumed traction-free. But, unlike J_1, it is not clear that the first term is necessarily zero on Γ_t and Γ_b. It would appear that in order to produce a contour independent result we need to include the parts of the path along the upper and lower crack faces. But actually, in practice, this may not be necessary. Consider the explicit form of the LEFM fields for arbitrary 2D loading, i.e. an arbitrary combination of Mode I and Mode II. It turns out that W is the same at corresponding points on the top and bottom surface of the crack. So, at least near the crack tip, and for the linear elastic case, the contributions from Γ_t and Γ_b cancel. (NB: this is because the two paths run in opposite directions). In fact, the same is found to be true for the elastic-plastic crack tip fields, the HRR fields.

Hence, path independence of the J_2 integral may be found, in practice, even without explicitly including the contributions from the parts of the contour along the crack faces - though this is not a strict algebraic result for finite sized contours. In addition, we may wish to start and finish the contour Γ_1 at different positions along the crack front, in which case the Γ_t and Γ_b integrals will not cancel. So, we write in general,

$$J_2 = -\int_{\Gamma_1}\left[Wdx + \overline{T}\cdot\frac{\partial\overline{u}}{\partial y}ds\right] - \int_{\Gamma_t+\Gamma_b} Wdx \tag{23b}$$

but noting that the second term will have a vanishing contribution from near the crack tip due to cancellation. If we restrict Γ_1 to start and end at the same distance from the crack tip, and if we are willing to believe that the crack tip fields are indicative of the cancellation of the second term even at large distances, then we can make the approximation,

$$J_2 \approx -\int_{\Gamma_1}\left[Wdx + \overline{T}\cdot\frac{\partial\overline{u}}{\partial y}ds\right] \tag{23c}$$

Note that, as for J_1, we have defined: $J_2 = \dfrac{\Im_2}{t}$ (24)

Finally, consider i = 3 in (12). Substituting in (15) gives,

$$\Im_3 = \oiint_{\delta R}\left[W\delta_{3k} - \frac{\partial W}{\partial u_{j,k}}\cdot\frac{\partial u_j}{\partial z}\right]dS_k = \oiint_{\delta R}\left[W\delta_{3k} - \sigma_{jk}\cdot\frac{\partial u_j}{\partial z}\right]dS_k = 0 \tag{25}$$

Consider firstly the case of a 2D problem, such as plane stress, plane strain or engineering plane strain. The tractions on the end caps are either zero or in the z-direction. In the former case the second term in the integrand is zero on the ends caps. In the latter case the second term involves $\dfrac{\partial u_z}{\partial z} = \varepsilon_{zz}$, which is the uniform out-of-plane strain. It follows that, whilst the contribution of one end cap may be non-zero, it is equal and opposite to the contribution of the other end cap (since the direction of the surface element is reversed, i.e. $dS_z = dxdy$ on one end cap but $dS_z = -dxdy$ on the other). The same is true for the W term. Hence, overall the end caps make no contribution to the integral. Considering now the integrals over the in-plane boundary

Γ, the first term is clearly zero since $dS_z = 0$ on this boundary. But the second term is also zero because the in-plane displacements, u_x and u_y, do not vary in the out-of-plane direction, z, i.e. $\frac{\partial \overline{u}}{\partial z} = 0$. Hence we conclude that J_3 is identically zero for 2D problems of plane stress, plane strain or engineering plane strain. Surmising that J_3 is related to the Mode III stress intensity factor (to be shown later), this is hardly surprising.

It remains, therefore, to consider J_3 in the case $u_x = u_y = 0$ and with u_z dependent upon (x,y) but constant in the z-direction, i.e. anti-plane strain. The only non-zero strains, and hence stresses, are the xz and yz shear components. It follows that the in-plane boundary Γ gives no contribution to the integral. Note that the second term is zero because $\frac{\partial u_z}{\partial z} = 0$

The first term in the integrand cancels out between the two ends caps, for the same reason as before. The second term involves dS_z and hence either σ_{zx} or σ_{zy}, and hence either $\frac{\partial u_x}{\partial z}$ *or* $\frac{\partial u_y}{\partial z}$, both of which are zero. Hence, J_3 is also identically zero for the case of anti-plane strain loading.

So, is J_3 always zero? Or does it require some in-plane and some out-of-plane loading in order to be non-zero? It turns out that the latter is the case. This will be shown next.

Evaluation of J_1, J_2, J_3 for the LEFM Fields

We can shed some light on the meaning of this vectorial $\bar{J}$ by evaluating each component in the LEFM case. In principle this is simply accomplished by substituting the explicit LEFM expressions for the near-tip stresses, strains and displacements in terms of the three stress intensity factors, K_I, K_{II} and K_{III}, into the integrals for each component of $\bar{J}$ and then evaluating the integrals. The algebra is nasty, though. In 30 years I have never previously bothered to carry out this calculation, and I soon regretted not waiting another 30 years. I have now carried out this explicit integration, with the help of MAPLE. The LEFM stresses and displacements which enter the calculations have been given in polar coordinates in the notes to session 13. (There are some notes in the corresponding article on http://rickbradford.co.uk to assist anyone wanting to carry out the same exercise for themselves).

The first two components of the vector J evaluate to:-

$$EJ_1 = \lambda\left(K_I^2 + K_{II}^2\right) + (1+\nu)K_{III}^2 \qquad (26)$$

$$EJ_2 = -2\lambda K_I K_{II} \qquad (27)$$

where $\lambda = 1$ in plane stress and $\lambda = 1 - \nu^2$ in plane strain. But we already know from session 15 that the RHS of Equ.(26) is the energy release rate. So we have successfully recaptured the fact that $J_1 = G$, the energy release rate, in the elastic case. Of course, we know from session 15 that J_1 is actually equal to the energy release rate even in the elastic-plastic case (for monotonic loading).

Equ.(27), however, is new. This shows that J_2 is zero unless *both* K_I and K_{II} are non-zero. In the case of 2D problems (i.e., when $K_{III} = 0$), knowledge of J_1 and J_2 suffices to deduce K_I and K_{II}, apart from their absolute sign (which is generally obvious).

Consequently, evaluation of both in-plane components of J is a means of separating the two SIF modes, K_I and K_{II}.

There remains the J_3 integral. As we observed above, this can only be non-zero if there are both in-plane and out-of-plane loadings. Suppose the loading is a combination of a 2D loading (plane stress or strain) together with anti-plane shears. J_3 reduces to,

$$J_3 = -\int_{\Gamma_1} \overline{T} \cdot \frac{\partial \overline{u}}{\partial z} ds = -\int_{\Gamma_1} T_z \cdot \frac{\partial u_z}{\partial z} ds = -\int_{\Gamma_1} T_z \cdot \varepsilon_z ds \qquad (28)$$

noting that the end caps again give zero net contribution. Thus if the nature of the loading ensures that there is no out-of-plane strain, then J_3 is zero. However, if the 2D part of the loading approximates to plane stress, then the z-strain is just $-\nu(\sigma_x + \sigma_y)$, and the integral evaluates in the LEFM case to,

$(\sigma_z = 0,\ \sigma_{xz} \neq 0, \sigma_{yz} \neq 0)$: $\quad EJ_3 = -\nu K_{II} K_{III}$ (29a)

(Plane strain): $\quad EJ_3 = 0$ (29b)

WARNING: I have failed to find a discussion of J_2 and J_3 in the standard texts. Consequently be cautious about the accuracy of the above remarks.

Qu.: Is $\bar{J}$ really a vector, or is it just three numbers?

It really is a vector, because ξ_{ij} in Equ.(12) really is a tensor.

Qu.: Can J_2 and J_3 be interpreted as energy release rates for growth at an angle?

Consider a non-self-similar extension of a crack at a non-zero angle to the x-axis, say by a vector increment, $\delta\bar{a} = (\cos\theta, \sin\theta)\delta a$. It is perfectly reasonable to define an energy release rate, G_θ, corresponding to this 'kinked' crack extension. Since $\bar{J}$ is a vector, it is very tempting to think that the energy released by the extension $\delta\bar{a}$ is just $\bar{J} \cdot \delta\bar{a}$, so that we would have $G_\theta = \bar{J} \cdot \hat{n}$, where $\hat{n} = (\cos\theta, \sin\theta)$. This is exactly what ABAQUS evaluates when "VCE" is used, i.e., it evaluates the J-contour for a virtual crack growth at a User input angle. The ABAQUS manual gives every impression that the result of this "VCE" J integral estimation is G_θ, the energy release rate for a kinked extension.

But it is not. I believe the theorists behind ABAQUS got it wrong (or at least introduced an undeclared approximation).

Qu.: So what does $\bar{J} \cdot \delta\bar{a}$ mean then?

The method that was used in session 15 to show that J_1 equals the energy release rate for self-similar growth ($\theta = 0$) can be repeated for an arbitrary angle of growth. The result is that $\bar{J} \cdot \delta\bar{a}$ must be identified with the energy released when the ***whole of the crack within the contour is displaced*** by $\delta\bar{a}$. The crack thus develops a dog-leg of size $\delta\bar{a}$ at the contour position. This does not correspond to the desired kinked extension at the crack tip.

However, the two may become the same thing if the contours start and finish near the crack tip. We have already seen that starting and finishing the contour near the crack tip is desirable to eliminate the second term in Equ.(23b). As long as this is done, the dog-leg is near the crack tip and may (we surmise) approximate the desired kinked extension.

It may be that the domain integral formulation used within ABAQUS addresses this automatically - I have not checked.

Qu.: Should we use ABAQUS VCE ?

Yes, do use it.

It is still sensible to look at how $\bar{J} \cdot \hat{n}$ varies with angle of crack extension – and to use the maximum value in assessments – even though the interpretation of this quantity in terms of energy release rate is suspect in my view.

APPENDIX

Relationship between conservation and vanishing divergence

Qu. How is conservation formulated? Integral form…

Suppose the density of this conserved stuff at position $\bar{r}$ and time t is $\rho(\bar{r},t)$. The total amount of stuff anywhere in space at time t is $\int \rho dV$, so this quantity must be constant, i.e., $\frac{d}{dt}\int \rho dV = 0$. But this is not enough. This equation could be true even if we had a big chunk of stuff suddenly vanishing from some region $\Re$, providing that an equal total amount of stuff magically materialised in other places at the same time. On the other hand, we don't want to insist that $\frac{d}{dt}\int \rho dV = 0$ is true for integration over any arbitrary region either – because this would imply $\frac{d\rho}{dt} = 0$, which would stop any stuff moving around. But we really want this stuff to be able to move around.

What we mean by conservation is that, for an arbitrary region, $\Re$, the decrease in the amount of stuff within it is balanced by the amount flowing out through its boundary, $\partial\Re$. So, to formulate conservation it is not enough to talk only about the density of the stuff, $\rho(\bar{r},t)$. We also need to know how the stuff is flowing. All we need to do this is the velocity of the stuff at every point, $\bar{v}(\bar{r})$. The flux of stuff is simply $\bar{q} = \rho\bar{v}$, where 'flux' means the amount of stuff which flows per unit time through unit area whose normal is parallel to the velocity. Conservation is then expressed as,

$$-\frac{d}{dt}\int_{\Re} \rho dV = \oiint_{\partial\Re} \rho\bar{v}\cdot d\bar{A} \qquad (1)$$

Equ.(1) is required to hold for an arbitrary choice of the region $\Re$, however large or small. The LHS of (1) is the rate of decrease of the amount of stuff within the region $\Re$. The RHS of (1) is the rate at which stuff flows out of the surface of $\Re$. Providing there are no sources or sinks of the stuff, these two things must be equal if stuff is conserved.

> The integral formulation of the conservation of stuff is: $-\frac{d}{dt}\int_{\Re} \rho dV = \oiint_{\partial\Re} \rho\bar{v}\cdot d\bar{A}$

Qu. How is conservation formulated? Differential form…

By the divergence theorem we can write the RHS of (1) as,

$$\oiint_{\partial\Re} \rho\bar{v}\cdot d\bar{A} \equiv \int_{\Re} \bar{\nabla}\cdot(\rho\bar{v})dV \qquad (2)$$

Now equating the LHS of (1) with (2), and noting that $\Re$ can be any arbitrary region, means that we must have,

$$-\frac{\partial\rho}{\partial t} = \bar{\nabla}\cdot(\rho\bar{v}) \qquad (3)$$

This is the differential (or local) form of the conservation condition. (1) and (3) are equivalent.

The local formulation of the conservation of stuff is: $-\frac{\partial \rho}{\partial t} = \overline{\nabla} \cdot (\rho \overline{v}) = \overline{\nabla} \cdot \overline{q}$

Qu.: No, I still don't get how this becomes path independence of J

What we have so far is that homogeneity tells us that there is some quantity whose density is ρ and whose flux is $\overline{q}$ and which is conserved, and that this means $-\frac{\partial \rho}{\partial t} = \overline{\nabla} \cdot \overline{q}$. But what are we left with when the situation is static is simply $\overline{\nabla} \cdot \overline{q} \equiv q_{k,k} = 0$. This is the residual effect in the static case of what would be a conserved 'current' in the dynamic case. But if we have a vector whose divergence is zero then $\oiint_{\partial \Re} \overline{q} \cdot d\overline{A} = 0$ over any closed surface $\partial \Re$ because the divergence theorem says this must equal $\iiint_{\Re} \overline{\nabla} \cdot \overline{q} dV$ which is identically zero if $\overline{\nabla} \cdot \overline{q} \equiv 0$. But if we defined the closed surface $\partial \Re$ as prismatic, and assuming the integrals over the 'end caps' are zero or can be made negligible, then the surface integral $\oiint_{\partial \Re} \overline{q} \cdot d\overline{A} = 0$ reduces to a contour integral for a 2D problem. So that's how Noether's conservation theorem produces a contour independent integral.