SQEP Tutorial Session 15: T73S02

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Post-yield fracture mechanics (PYFM): why LEFM is not adequate; Definition of energy release rate; PYFM criterion for 'fracture' (tearing); Graphical meaning of energy release rate: distinction between strain energy and complementary energy; Derivation of the LEFM energy release rate in terms of the Ks; Effective K for multi-mode loading; Proof that the contour integral for J equals the energy release rate, G; Limitations for cyclic loading/reversed plasticity.

Qu.: Why is LEFM of limited use for structural steels?

LEFM is of limited use for structural steels because these are generally ductile and exhibit plasticity prior to fracture. The basis of LEFM is the existence of the LEFM fields in some region surrounding the crack tip. If the plastic zone is widespread, perhaps across the whole section, then an LEFM region will probably not exist.

Q.: So if there is any yielding at all, LEFM is no good?

Not so.

LEFM may still be valid, i.e., the fracture criterion may still be that $K_I = K_{Ic}$, even if there is plasticity around the crack tip. As long as there is a region surrounding the crack tip in which the stresses and strains are given by the LEFM fields, then the corresponding K must control fracture. This is because whatever is going on inside the process zone – possibly large plastic strains – it is controlled by the LEFM fields which act as a boundary condition.

Qu.: But doesn't even a contained yield zone modify the effective K a bit?

Yes.

Even if the yield zone is small compared with the structural dimensions, the effect of the crack tip plasticity can be to displace the effective centre of the LEFM fields - as if the crack were of a slightly longer length, $a \to a' = a + \Delta a_p$. This results in the "small scale plasticity correction" to K. This is discussed further in a later session in the context of Failure Assessment Diagrams.

Qu.: So, if the ***structural*** dimensions are large compared to the yield zone, LEFM is still useful?

Yes and no.

If the structural dimensions are large compared to the yield zone, then the LEFM fracture criterion $K_I = K_{Ic}$ may be valid for the ***structure***. The snag is, what value does the toughness, K_{Ic}, take? The chances are that the test specimens used in the lab to measure the toughness are much smaller than the plant component. The toughness tests may therefore involve yield zones which are large compared with the specimen size. So we need post-yield fracture mechanics (PYFM) to devise a valid means of measuring the toughness from the lab specimens anyway, even if the structure which is being assessed is sufficiently close to LEFM.

Qu.: So, when is PYFM required?

PYFM is required if, for either the structure OR the lab specimen used to measure the toughness, there is no region around the crack tip in which the fields are LEFM.

But there is another reason for PYFM. In linear elastic fracture mechanics it is assumed that $K_I = K_{Ic}$ precipitates fast fracture. But ductile materials tend to exhibit stable crack extension (tearing) prior to complete fracture. So PYFM is also required to distinguish between stable tearing and fast fracture, and to provide a methodology for assessing stable tearing.

Qu.: Are there "PYFM crack tip fields" analogous to the LEFM fields?

Yes and no.

Under linear elastic conditions, the LEFM fields are completely general. They always apply sufficiently close to a sharp crack tip. There are no elastic-plastic fields of equivalent generality.

One reason is the irreversibility of plasticity – which means that (roughly speaking) for a given stress state the total plastic strain can be contrived to be almost anything. So clearly there is no unique plastic strain field.

But even if we consider only monotonically increasing load (so that irreversibility is irrelevant) then different structural geometries or different types of loading will often produce different crack tip fields once sufficiently large plastic strains have spread across the whole section. This is because the crack tip region becomes overwhelmed by the global structural response, i.e., the collapse mechanism. This is known as "loss of constraint" or "loss of validity".

However, there are a set of PYFM crack tip fields which apply in restricted circumstances, known as the HRR fields (for Hutchinson and Rice & Rosengren who developed the theory in 1968). They apply only for power-law hardening materials. They are defined as the unique elastic-plastic crack tip fields which occur for a given power law hardening if the plastic zone is surrounded by an LEFM region.

However, the HRR fields may continue to apply near the crack tip even when there is no LEFM region – and perhaps even when plasticity has spread across the whole section. But, unlike the LEFM case, the crack tip fields cannot be assumed always to be of HRR form. Just as LEFM is valid if there is a region surrounding the crack tip in which the LEFM fields prevail, so PYFM is valid if there is a region surrounding the crack tip in which the HRR fields prevail (when power law hardening is assumed).

The HRR fields are discussed in a later session. For now we don't need them because there is an alternative way to express the PYFM fracture criterion…

Qu.: What do we mean by "energy release rate" in fracture mechanics?

Energy release rate, G, is defined as the energy provided by the strain energy of the body and/or the externally applied forces when the crack size is increased by a unit increment of crack area. More accurately, G is the derivative of energy with respect to crack area (the term 'rate' does not imply anything changing in time).

Note that in 3D the energy release rate will depend upon where along the crack front the increment of area is placed.

Qu.: Do we count both crack faces or only one in the definition of crack area?

By convention, just one.

Qu.: Why is energy release rate of interest?

G is of interest because it provides another means of stating the fracture criterion. We may postulate that cracks in a given material require a 'critical' amount of energy to be supplied to the crack tip region (the process zone) in order for the crack to advance by unit area. The fracture criterion is then $G = G_c$.

Note that this fracture criterion does not rely upon any particular crack-tip field, unlike the K-based criterion, $K = K_{Ic}$, for which K is defined only with respect to the LEFM fields. This means that the criterion $G = G_c$ can be extended to apply also in post-yield fracture, where the crack tip fields are elastic-plastic.

Qu.: How do we reinterpret G for elastic-plastic conditions?

For proportional loading, elastic-plastic behaviour is equivalent to non-linear elasticity. The above definition of energy release rate will still apply for non-linear elasticity. It therefore applies also for elastic-plastic behaviour under proportional loading, but with 'strain energy' reinterpreted to mean the elastic-plastic work integral, $U \to \int \left(\int \sigma_{ij} d\varepsilon_{ij} \right) dV$. This reinterpretation may be expected to be reasonable so long as the loading is monotonically increasing. But if stresses reduce, post-yield fracture mechanics becomes problematic (in principle). This is discussed in a later session but will be ignored here.

For convenience of exposition we continue to refer to U as 'strain energy', as it would be for non-linear elasticity, even though we are really dealing with elastic-plastic material (for which part of the elastic-plastic work is irrecoverable).

Aside on notation: Most authors reserve "G" for the energy release rate under linear elastic conditions, preferring the symbol "J" for the energy release rate under elastic-plastic conditions. I have preferred to use "G" for the energy release rate whether the conditions are elastic or elastic-plastic. I reserve the symbol "J" to mean a quantity calculated using a contour integral (later). Where PYFM is valid, we will see that $J = G$ in any case, so the symbol indicates only the method which is used to calculate the quantity, not its value (which is the same).

Qu.: How can G be calculated?

If no energy is supplied by any external agency to the body whilst the crack is extending, it follows that the energy requirement for the crack extension can be supplied only from the body's strain energy, U.

Saying that no external agency does work on the body as the crack extends is the same as saying that all the applied 'loads' are actually applied displacements or applied strains. As the crack extends, the body becomes less stiff – so the loads corresponding to the constant displacements reduce. This means that the strain energy of the body reduces.

Hence, in the case of applied displacements only, the energy release rate is given by $G = -\frac{\partial U}{\partial A}$, where U is the total strain energy of the body.

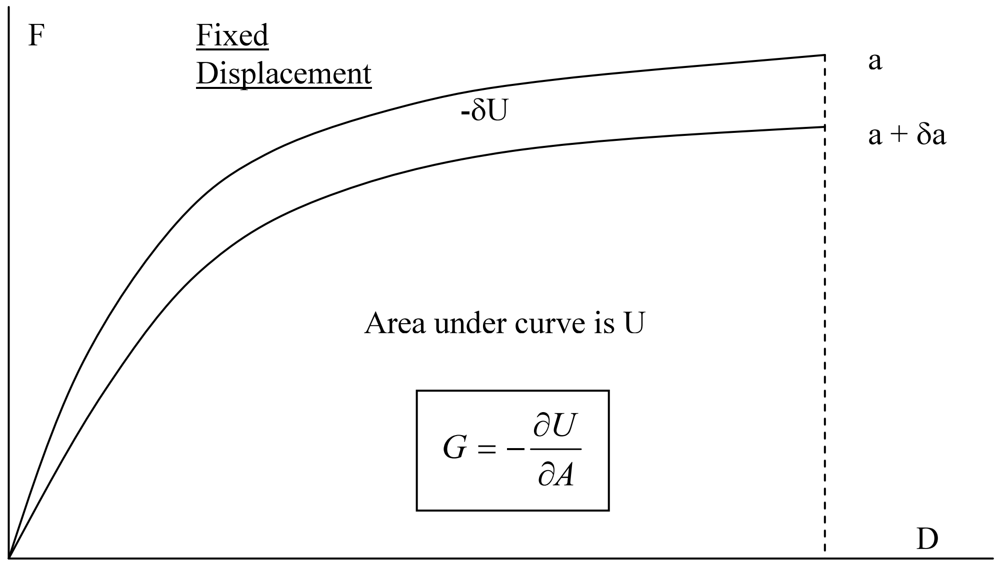
In a 2D case, if the out-of-plane thickness is t and the crack extends by da, then $dA = tda$ and $G = -\frac{1}{t}\frac{dU}{da}$.

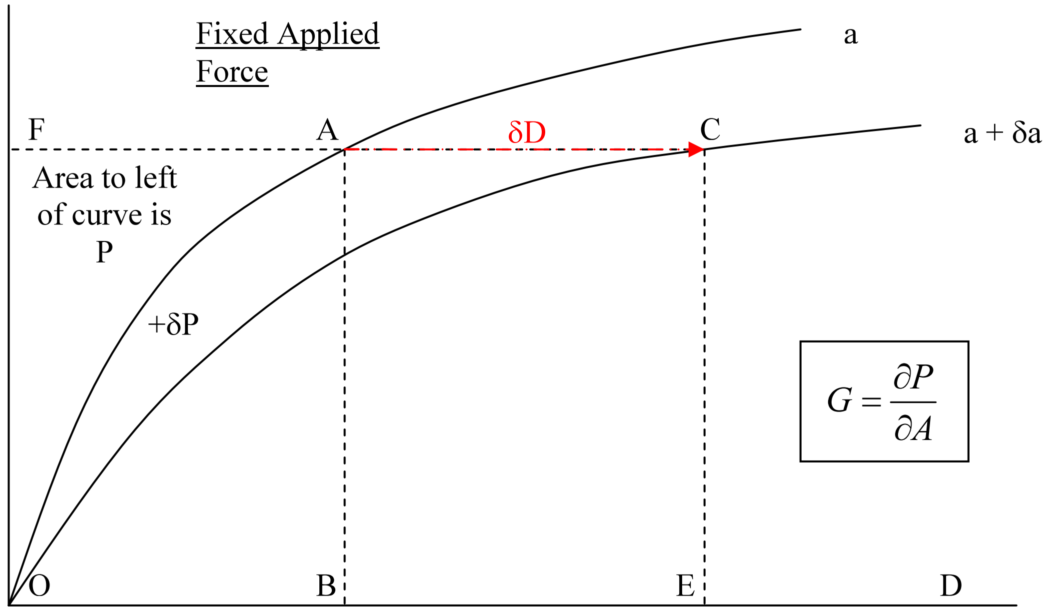
Qu.: OK, but what if there are applied loads?

If there are applied loads, as well as applied displacements, acting on the body, then the external agency does work when the crack extends (because the increased compliance of the structure leads to movement at the point of application of the loads). Moreover, if the applied loads dominate over the applied displacements, the strain energy of the body will actually increase during crack extension. So, the strain energy of the body cannot now be the source of the energy required to drive the crack (because the strain energy does not decrease).

In this case the energy for crack extension is supplied by the external agency. Since the external agency also supplies the energy to increase the body's strain energy, the energy available for crack advance is the work done by the external agency less the increase in the strain energy, $G = \frac{\partial}{\partial A}(\mathrm{W.D.} - U)$.

For the general case, with mixed types of loading, a potential energy is defined: $P = \sum_{\text{applied loads}} F_i D_i - U$, where F_i are the loads which are constant during crack extension (i.e. the load-controlled loads) and D_i are the displacements at their point of application. Hence, the general definition of energy release rate is $G = \frac{\partial P}{\partial A}$.





Strain energies: U(a) = OAB and U(a+δa) = OCE
Hence, δU = U(a+δa) - U(a) = ACEB - δP = F.δD - δP
(ignoring quantities of second order)

Qu.: In what direction is the increment of crack growth?

Unless otherwise stated, the crack is assumed to extend in a self similar direction, i.e., on $\theta = 0$.

However, it is perfectly reasonable to define an energy release rate for crack growth at angle θ – call it G_θ. If the energy release rate is greatest for some non-zero θ, then it is advisable (conservative) to employ this larger value in assessments. This will only be the case if the shear modes are significant.

It is tempting to assume that a crack will tend to grow in the direction of maximum energy release rate. This may be so, but not necessarily. The direction of crack growth can be problematic. It may depend upon whether the fracture mechanism is more ductile or more brittle. The orientation of the maximum principal stress may be a controlling influence, rather than G_θ. Or, if beyond general yield, the global plastic strains (slip lines) may be the controlling factor. Moreover, for very small amounts of growth, the local grain structure will dominate.

Qu.: Does the ABAQUS J-Contour integral VCE method calculate G_θ?

ABAQUS gives the impression that it does. But I suspect it may not. This is discussed in a later session. (There is no problem with *J*=*G* on $\theta = 0$, however).

Summary of Formulae for G

(1) General Case: $G = \frac{\partial P}{\partial A}$ elastic-plastic

(2) Displacement Control: $G = -\frac{\partial U}{\partial A}$ elastic-plastic

(3) Elastic Load Control: $G = +\frac{\partial U}{\partial A}$ elastic

For combined loading types, only (1) is valid, even for elasticity.

Case (3) is a fluke which comes about because $\Delta P = \Delta U$ in linear elasticity and load control.

Fracture Criterion

$$G = G_c$$

…but note that 'fracture' in PYFM may mean stable tearing rather than fast fracture,

…and note that it is implicitly assumed that the out-of-plane constraint is the same as that in the test which measures G_c.

Qu.: But how does this PYFM fracture criterion relate to the LEFM criterion?

In the elastic case, the fracture criterion $G = G_c$ must reduce to the LEFM criterion $K_I = K_{Ic}$. To demonstrate this we need to find G in terms of K in the elastic case…

The LEFM fields show that the stresses directly ahead of a crack tip (on θ = 0) under mixed mode loading are,

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}, \quad \tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}, \quad \tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \tag{1}$$

The displacements of the upper crack face (θ = π) are,

$$u_y = 4\kappa \frac{K_I}{E}\sqrt{\frac{r}{2\pi}}, \quad u_x = 4\kappa \frac{K_{II}}{E}\sqrt{\frac{r}{2\pi}}, \quad u_z = 4(1+\nu)\frac{K_{III}}{E}\sqrt{\frac{r}{2\pi}} \tag{2}$$

where $\kappa = 1$ in plane stress and $\kappa = 1 - \nu^2$ in plane strain.

Imagine tractions to be applied to the crack faces near the tip in order to cause the crack to close over a length Δ from the original tip. Clearly the tractions are simply given in terms of the stresses, (1), ahead of the new (smaller) crack, at a distance $r' = \Delta - r$. As long as the crack closure is infinitesimal, the SIFs can be taken as unchanged. Assuming elastic behaviour, the work done by the tractions in closing a length δr of the crack tip region is simply half of the product of the corresponding tractions and displacements:-

$$\delta \mathrm{WD} = \frac{t\delta r}{2}\left\{\sigma_y(r')u_y(r) + \tau_{xy}(r')u_x(r) + \tau_{yz}(r')u_z(r)\right\} \times 2 \tag{3}$$

where the factor of 2 accounts for the two crack faces, and t is the thickness in the crack front direction (z). Substituting (1) and (2) into (3) and integrating to find the total work done to close a finite length Δ of the crack gives,

$$\mathrm{WD} = t\int_0^{\Delta} dr' \left\{ 4\kappa \frac{K_I^2}{2\pi E}\sqrt{\frac{r}{r'}} + 4\kappa \frac{K_{II}^2}{2\pi E}\sqrt{\frac{r}{r'}} + 4(1+\nu)\frac{K_{III}^2}{2\pi E}\sqrt{\frac{r}{r'}} \right\} \tag{4}$$

The required integral can be evaluated using the substitution $r' = \Delta \cos^2 \theta$. This gives,

$$\int_0^{\Delta} \sqrt{\frac{r}{r'}} dr' = \frac{\pi}{2}\Delta \tag{5}$$

So that (4) becomes simply,

$$\frac{WD}{t\Delta} \equiv G = \kappa \frac{K_I^2}{E} + \kappa \frac{K_{II}^2}{E} + (1+\nu)\frac{K_{III}^2}{E} \tag{6}$$

This is the required expression which gives the energy release rate, G, in terms of the stress intensity factors.

Qu.: So, what is the critical value of energy release rate, G_c ?

A properly constrained (hence valid) fracture toughness test can be taken to be in plane strain, and hence we get,

$$G_c = \left(1-\nu^2\right)\frac{K_{Ic}^2}{E} \tag{7}$$

For example, a fracture toughness of 100 MPa√m in a material with E = 200 GPa and ν = 0.3 gives G_c = 0.0455 MPa.m = 45.5 N/mm = 0.0455 J/mm^2 = 45.5 kJ/m^2.

Qu.: What is the "effective stress intensity factor"?

Equ.(6) gives rise to an effective SIF in mixed mode loading, defined as,

Plane Stress: $$K_{eff} = \sqrt{K_I^2 + K_{II}^2 + (1+\nu)K_{III}^2} \tag{8a}$$

Plane Strain: $$K_{eff} = \sqrt{K_I^2 + K_{II}^2 + \frac{K_{III}^2}{(1-\nu)}} \tag{8b}$$

NB: This K_{eff} should not be confused with the effective SIF range sometimes employed in fatigue crack growth assessments.

Qu.: What are the shear mode toughnesses?

If $G = G_c$ were the correct fracture criterion, then the mixed mode fracture criterion in elastic conditions would be $K_{eff} = K_{Ic}$. This implies that the Mode II toughness is the same as the Mode I toughness, whilst the Mode III toughness is a factor $\sqrt{1-\nu}$ smaller. However, these conclusions are questionable. I suspect there is little experimental evidence to support this. I would expect the shear mode toughnesses to exceed that in Mode I, because of the reduced hydrostatic stress ahead of the crack in the shear modes. Consequently, in mixed mode fracture, I suspect that the total energy release rate (or equivalently, K_{eff}) will be a conservative parameter to use in assessments.

Qu.: What is J?

J is a fracture parameter defined via a contour integral. It does not provide yet another fracture criterion because J and G are numerically the same, at least for proportional loading (non-linear elasticity). So J is also the energy release rate, but calculated in a different manner.

A derivation of the J-integral will be given in detail in a later session. Here we just state what it is and indicate what the terms mean and why it equals the energy release rate.

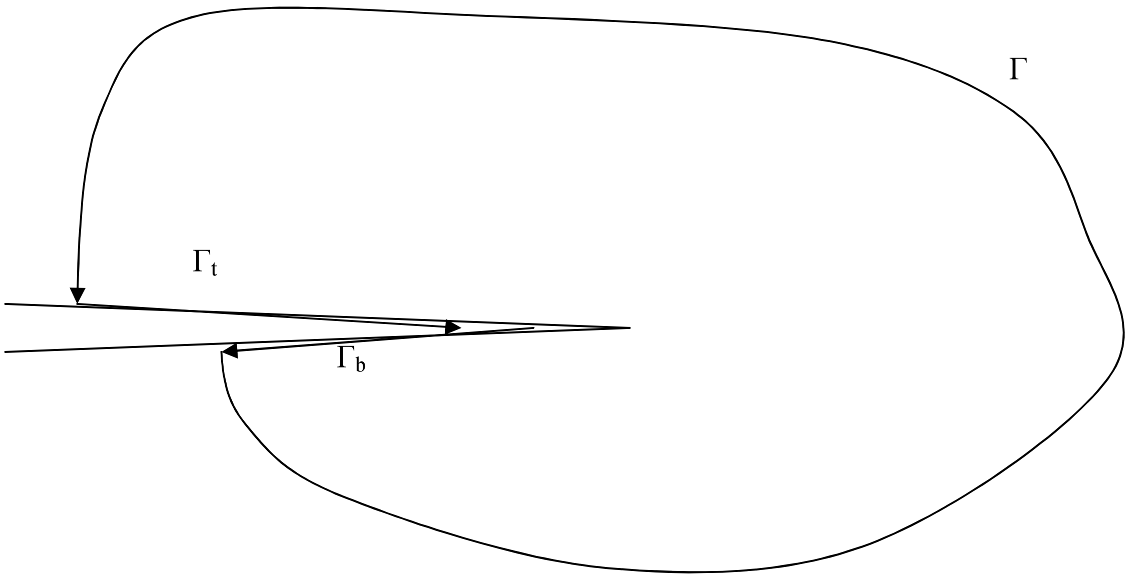
J for self-similar growth

$$J = \int_{\Gamma} \left[W dy - \bar{T} \cdot \frac{\partial \bar{u}}{\partial x} ds \right] \tag{9}$$

where, ds is the element of length around a contour Γ surrounding the crack tip and beginning and ending on the lower and upper crack faces (see diagram).

$W = \int \sigma_{ij} d\varepsilon_{ij}$ is the elastic-plastic work per unit volume (the strain energy density for a non-linear elastic material), $\bar{T}$ is the traction acting over the contour Γ, and $\bar{u}$ is the displacement field.

Note that it does not matter whether we include additional parts to the contour which run along the crack faces, i.e., Γ_t and Γ_b (see diagram) – **so long as the crack faces are straight!** This is because they contribute nothing, since $dy = 0$ and $\bar{T} = 0$ on the crack faces.



Proof that $J = G$

Suppose the crack suffers a small displacement, $\bar{\Delta}$, in the x-direction. We wish to evaluate the energy which flows into the defect during the crack movement. Consider the boundary $\tilde{\Gamma}$ defined by displacing Γ by $\bar{\Delta}$. Suppose that the tractions and displacements on the boundary Γ are initially T and u . And suppose that, after the displacement of the crack, the tractions and displacements on the displaced boundary, $\tilde{\Gamma}$, are $\tilde{T}$ and $\tilde{u}$. The correct physical situation can be reached in two notional steps:-

[1] At the initial crack size, consider the region within Γ to be cut out from the body, but holding the tractions on Γ fixed. Consider the region $\tilde{\Gamma}$ to be cut out of a copy of the body after crack extension. Then the initial cut-out Γ will drop neatly into place in the hole left by the removal of $\tilde{\Gamma}$ from the second body after the crack displacement. There will be a slight misfit due to the tractions on Γ and $\tilde{\Gamma}$ being different. But this is remedied next.

[2] The second step is to adjust the tractions acting on the material within the boundary so that $T \to \tilde{T}$, which leads also to $u \to \tilde{u}$.

Step [1] is the initial crude approximation. Step [2] makes this notional process precise.

How much would the strain energy within Γ change if step [1] were accurate? In the approximation of step (1), the strain energy density at point $\bar{r}$ after the crack displacement is just $W(\bar{r}-\bar{\Delta})$, where $W(\bar{r})$ is the strain energy density at the same point before the crack movement. Hence, for step [1], the strain energy change within the boundary Γ is just,

$$\Delta E_1 = t\int_V W(\bar{r}-\bar{\Delta})dxdy - t\int_V W(\bar{r})dxdy \tag{10}$$

where V is the volume within Γ and t is the thickness of the 2D slice of material being considered. But,

$$W(\bar{r}-\bar{\Delta}) - W(\bar{r}) = -\Delta\frac{\partial W}{\partial x} \tag{11}$$

because $\bar{\Delta}$ is in the x-direction. Hence, (10) becomes,

$$\Delta E_1 = t\int_V \left[W(\bar{r}-\bar{\Delta}) - W(\bar{r})\right]dxdy = -t\Delta\int_V \frac{\partial W}{\partial x}dxdy = -t\Delta\oint_\Gamma W dy \tag{12}$$

So that,

$$\frac{\Delta E_1}{t\Delta} = -\oint_\Gamma W dy \tag{13}$$

The RHS of (13) is just the first term in the definition of J, Equ.(9), whilst the LHS is the contribution of Step [1] to the energy release rate (since $t\Delta$ is the increment of crack area). So far so good. Now we need to look at Step [2], which makes the process exact.

In Step [2] the adjustment of the tractions, or more precisely the corresponding adjustment of the boundary displacements, causes Γ to do work on the surroundings. Per unit surface area of $\tilde{\Gamma}$ the change in the strain energy due to this is approximately,

$$-\frac{1}{2}\left(\tilde{\bar{T}} + \bar{T}\right)\cdot\left(\tilde{\bar{u}} - \bar{u}\right) \approx -\bar{T}\cdot\left(\tilde{\bar{u}} - \bar{u}\right) \tag{14}$$

and this becomes precise in the calculus limit of small crack displacements. But at the end of step [1] we have not achieved the correct displacements $\tilde{\bar{u}}$. But the corrected displacements are given by the initial field $\bar{u}$, but evaluated at the point $\bar{r}-\bar{\Delta}$. Hence,

$$\tilde{\bar{u}}(\bar{r}) \approx \bar{u}(\bar{r}-\bar{\Delta}) \approx \bar{u}(\bar{r}) - \Delta\frac{\partial \bar{u}}{\partial x} \tag{15}$$

Hence, $\tilde{\bar{u}}(\bar{r})-\bar{u}(\bar{r})\approx -\Delta\frac{\partial\bar{u}}{\partial x}$ and this provides an additional energy per unit area of $-\bar{T}\cdot\left(\tilde{\bar{u}}(\bar{r})-\bar{u}(\bar{r})\right)=\bar{T}\cdot\left[\Delta\frac{\partial\bar{u}}{\partial x}\right]$. So the increase in the strain energy within Γ due to Step [2] is just $\bar{T}\cdot\left[\Delta\frac{\partial\bar{u}}{\partial x}\right]$ per unit area of the boundary Γ, or a total energy of,

$$E_2 = t\Delta\oint_{\Gamma}\bar{T}\cdot\frac{\partial\bar{u}}{\partial x}ds \qquad (16)$$

Overall, the total strain energy increase within Γ, adding (13) and (16), is,

$$-t\Delta\oint_{\Gamma}\left[Wdy-\bar{T}\cdot\frac{\partial\bar{u}}{\partial x}ds\right] \qquad (17)$$

But the energy release rate is minus this energy change divided by the increment of crack area, which is just $t\Delta$. Hence,

$$G=\oint_{\Gamma}\left[Wdy-\bar{T}\cdot\frac{\partial\bar{u}}{\partial x}ds\right]=J \qquad \textbf{\underline{QED}}. \qquad (18)$$