SQEP Tutorial Session 13: T73S02

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The three modes of K; Why the LEFM fields vary as 1/√r; Derivation of $K = \sigma\sqrt{\pi a}$ *and* $\sigma = K/\sqrt{2\pi r}$ *illustrated for Mode III; The LEFM crack tip fields; Fracture toughness and the brittle fracture criterion.*

Everything in the notes for this session is for elastic behaviour (LEFM)

Qu.: What is the stress at the tip of a notch with root radius ρ?

The stress concentration factor (SCF) of a notch increases as the radius, ρ, decreases.

Qu.: So what does the notch tip stress become as the notch becomes sharp?

The notch tip stress diverges. The Inglis solution for an elliptical hole an infinite plate subject to uniaxial tension gives the SCF at the notch tip to be,

$$\text{SCF} = 1 + 2\sqrt{\frac{a}{\rho}}$$

where a is the semi-major axis and ρ is the radius of curvature at the pointy end of the ellipse, i.e. $\rho = b^2 / a$, where b is the semi-minor axis. Hence, the ellipse tends to an embedded crack of length $2a$ in the limit that $b, \rho \to 0$. So the elastic stress at the tip of a crack is infinite.

Qu.: If the remotely applied load is very, very small, what is the crack tip stress?

Infinite.

Qu.: So why doesn't anything with a crack in it break immediately?

Because the assumptions of a perfect continuum which is isotropic, homogeneous, with only small-strains, and only elastic behaviour break down – and so the stress isn't really infinite.

Qu.: Which of the assumptions breaks down: (a)continuum, (b)isotropic, (c)homogeneous, (d)small-strain, (e)elastic?

All of them.

Qu.: So should we just give up now, then?

Don't be so defeatist! We'll return to this later.

At least this makes it clear why we're concerned about cracks. They have a tendency to grow or to fast fracture, because of their attendant high crack tip stress fields.

Qu.: How do we describe the stresses near to the crack tip?

In terms of the stress intensity factor (SIF), K, the stress ahead of a crack loaded in Mode I is $\sigma_y(r) = \frac{K_I}{\sqrt{2\pi r}}$.

Qu.: What is the magnitude of the SIF in a simple case?

For an embedded crack of length $2a$ in an infinite plate subject to uniform uniaxial tension σ_∞ the SIF is given by $K_I = \sigma_\infty\sqrt{\pi a}$.

Qu.: Is there anything fundamental about those factors of π inside the √ ?

No. They are just a convention. The stress ahead of the crack is $\sigma_y(r) = \sigma_0\sqrt{\frac{a}{2r}}$, and we could equally have defined the SIF to be $K_I = \sigma_0\sqrt{a}$ and put $\sigma_y(r) = \frac{K_I}{\sqrt{2r}}$. In fact some authors do use this convention (e.g. some solutions quoted in Murakami), so you need to be careful when interpreting papers or you could get the SIF wrong by a factor of $\sqrt{\pi}$.

Qu.: Where does $K_I = \sigma_\infty\sqrt{\pi a}$ come from?

Of course this only holds so long as the crack is very small compared with the dimensions of the body. We'll look at the derivation from elasticity theory below. But for now we note that it follows simply if we are allowed to assumed that $G \propto K^2$ (proved in a later session). Here *G* is the energy release rate.

For primary loads, $G = \frac{\partial U}{\partial A}$ where *U* is the linear elastic strain energy and A is the crack face area (one of the two faces only). So, *G* is the increase in strain energy per unit increase in the crack area. A simple argument (given below) shows that, for very small cracks, *G* must be proportional to the crack length, *a*. Hence, if we are given that $G \propto K^2$, it follows that $K \propto \sqrt{G} \propto \sqrt{a}$. Moreover, since the stresses near the crack tip are proportional to *K*, it follows that *K* must be proportional to the applied loads in linear elasticity. Hence, we have $K \propto \sigma_\infty\sqrt{a}$. This completes the proof, noting that the factor of $\sqrt{\pi}$ just a convention.

To complete this demonstration we need only demonstrate that $G \propto a$ for very small cracks. This follows because, for a small crack, the opening displacement of its faces must be proportional to *a*, since *a* is the only dimension scale in the problem (when the structure can be taken as effectively infinite in comparison). Hence, the energy release due to the opening of a portion δa of the crack must be proportional to $a\sigma_\infty\delta a$, i.e., displacement times the element of force. Hence, the total energy release, ΔU, is proportional to $\int a\sigma_\infty\delta a \propto a^2$. Hence, the energy release rate, G, is $\propto \frac{\partial \Delta U}{\partial a} \propto a$.

QED.

Qu.: Why do the stresses vary as $1/\sqrt{r}$?

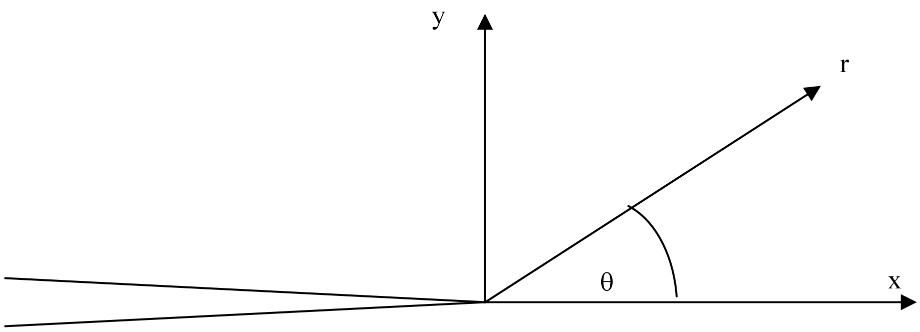
Argument #1: Once we know that $K_I = \sigma_\infty\sqrt{\pi a}$ it follows from dimensional analysis that the stress must be proportional to $K_I/\sqrt{r}$. QED. However, this is rather a cheat since to derive $K_I = \sigma_\infty\sqrt{\pi a}$, above, we had to assume $G \propto K^2$, and this will be proved in Session 14 by using the expressions for the crack tip fields, i.e., that the stresses are proportional to $1/\sqrt{r}$, so the argument is circular.

Argument #2: Appeal to the path independence of the J contour integral. The integrand of J is proportional to stress x strain, whilst the size of the contour produces a length scale dependence proportional to r. Hence $J \propto \sigma \cdot \varepsilon \cdot r =$ some constant, independent of contour. Since $\sigma \propto \varepsilon$ for elastic behaviour, it follows that $\sigma^2 r$ is a constant (independent of r), and hence that $\sigma \propto 1/\sqrt{r}$. This is a rigorous argument because the J integral and its contour independence can be proved without reference to the crack tip fields. But since we've not covered this yet, it's less than satisfactory at this stage.

Consequently, we shall derive $\sigma \propto K/\sqrt{r}$ directly, below. But first some definitions.

Qu.: What is the conventional coordinate system used in fracture mechanics?

It is conventional to position the crack along the negative x-axis, with the tip at the origin. The tensile stress which causes opening of the crack is then the y-stress.



Qu.: What are Mode I, II and III SIFs?

With reference to the above standard coordinate system, the three modes are defined by,

Mode I: The remotely applied stress is σ_{yy} (opening mode)

Mode II: The remotely applied stress is σ_{xy} (in-plane shear mode)

Mode III: The remotely applied stress is σ_{zy} (out-of-plane shear mode)

Qu.: What about the other three possible applied stresses, $\sigma_{xx}, \sigma_{zz}, \sigma_{xz}$?

They produce $K = 0$.

Qu.: A shaft under torsion has a part-penetrating, fully circumferential crack: what is the Mode?

Mode III

Qu.: A shaft under torsion has a part-penetrating, long semi-elliptic axial crack: what is the Mode at the deepest point?

Mode III

Qu.: A shaft under torsion has a fully-penetrating, axial crack: what is the Mode at the deepest point?

Mode II

Qu.: A shaft has a part-penetrating, fully circumferential crack: how could it be loaded to produce Mode II?

Apply a temperature difference to the shaft on either side of the crack.

Qu.: How is $\sigma \propto 1/\sqrt{r}$ deduced properly from elasticity theory?

Well, it's best not to start with Modes I or II, but to look at Mode III, because it's much simpler. This is anti-plane-strain in which the in-plane displacements are zero, $u_x = u_y = 0$, and the out-of-plane displacement, u_z, is independent of z, varying only with respect to x and y. Because the strains are given by $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, it follows that the only non-zero strains are the out-of-plane shear strains $\varepsilon_{xz} = u_{z,x}/2$ and $\varepsilon_{yz} = u_{z,y}/2$, and hence the only non-zero stresses are the out-of-plane shear stresses, $\sigma_{xz} = G\gamma_{xz} = Gu_{z,x}$ and $\sigma_{yz} = G\gamma_{yz} = Gu_{z,y}$. But the equilibrium equations, $\sigma_{ij,j} = 0$, reduce to just $\sigma_{zx,x} + \sigma_{zy,y} = 0$. Substituting for the stresses in terms of the displacements thus gives, $u_{z,xx} + u_{z,yy} = \nabla^2 u_z = 0$, where ∇^2 is the Laplacian operator in 2D, i.e., $\nabla^2 \equiv \partial_x^2 + \partial_y^2$. There follows a Mathematical Interlude…

Qu.: What is the general solution of $\nabla^2 \phi = 0$?

This is known as the harmonic equation, or Laplace's equation. The general solution is that any function which is analytic in the complex variable $s = x + iy$ will satisfy this equation. And vice-versa, any solution to $\nabla^2 \phi = 0$ must be an analytic function of $s = x + iy$. An "analytic" function is, roughly speaking, a function of s alone, rather than of x and y separately. It must also have a well defined complex derivative. The complex derivative, $\frac{d\phi}{ds}$, only makes sense if you get the same value for it whether you approach the point from the real or the imaginary direction. So an analytic function must obey $\frac{d\phi}{ds} = \frac{d\phi}{dx} = \frac{d\phi}{idy} = -i\frac{d\phi}{dy}$. But the derivative is also an analytic function, so taking the complex derivative again gives $\frac{d^2\phi}{ds^2} = \frac{d^2\phi}{dx^2} = -i\frac{d}{dy}\left(-i\frac{d\phi}{dy}\right) = -\frac{d^2\phi}{dy^2}$, which thus gives $\nabla^2\phi = 0$. Hence any analytic function satisfies Laplace's equation, as claimed.

Qu.: But does that mean that u_z in anti-plane strain can be any analytic function?

No, because we also have to satisfy the boundary conditions. The boundary conditions are,

[1] On the crack faces (the negative x-axis), the u_z displacement should be non-zero, but discontinuous over the crack, with equal and opposite values on the two crack faces;

[2] On the crack faces (the negative x-axis), all stress components except σ_{zx} must be zero, whilst σ_{zx} is equal and opposite on the two crack faces;

[3] The stresses will diverge as $r \to 0$, but not faster than $1/\sqrt{r}$ since otherwise the total strain energy near the crack tip would be divergent, which is unphysical.

Qu.: What is the simplest function which has the appropriate discontinuity over the crack?

In analytic function theory it is familiar that functions like $\sqrt{s}$ have a 'branch cut' over which they are discontinuous. This is just the generalisation of the fact that a square-root may be chosen to have either sign. It is conventional to place the branch cut along the negative x-axis. Putting $s = re^{i\theta}$ we thus have,

$$\sqrt{s} = \sqrt{r}e^{i\theta/2} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

If we approach the upper crack face from the positive y side ($\theta \to \pi$), then this gives $\sqrt{s} \to i\sqrt{r}$, whereas if we approach the lower crack face from the negative y side ($\theta \to -\pi$), then this gives $\sqrt{s} \to -i\sqrt{r}$. Hence, we respect boundary condition [1] if $u_z \propto \mathrm{Im}(\sqrt{s})$.

Qu.: Does $u_z \propto \mathrm{Im}(\sqrt{s})$ work then?

Let's see if it obeys the other boundary conditions. We note that the stresses are given simply by,

$$\sigma_{xz} \propto \mathrm{Im}\left(\frac{du_z}{ds}\right) = \mathrm{Im}\left(\frac{1}{2\sqrt{s}}\right) = \frac{1}{2\sqrt{r}}\mathrm{Im}\left(e^{-i\theta/2}\right) = -\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)$$

$$\sigma_{yz} \propto \mathrm{Im}\left(i\frac{du_z}{ds}\right) = \mathrm{Im}\left(\frac{i}{2\sqrt{s}}\right) = \frac{1}{2\sqrt{r}}\mathrm{Im}\left(ie^{-i\theta/2}\right) = \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$$

Hence, b.c. [2] is obeyed since $\sigma_{yz} = 0$ on $\theta = \pm\pi$, and all the other stress components are zero except $\sigma_{xz} \propto \mp 1/2\sqrt{r}$, equal and opposite on the two crack faces.

b.c. [3] is obeyed because the stress divergence is not faster than $1/\sqrt{r}$.

So, the above equations obey all the requirements for the elastic crack tip fields in Mode III, and display the claimed $1/\sqrt{r}$ dependence. But is this solution unique?

Qu.: Are there any other possible solutions?

There are no other solutions which obey the boundary conditions.

If, in $u_z \propto s^{\lambda}$, we tried a power $\lambda > 1$, then because the stresses are proportional to $r^{\lambda-1}$ they would be finite at the crack tip (and would be infinite as $r \to \infty$) and hence $\lambda > 1$ is not possible.

If we tried a power $\lambda < 0$, then because the stresses are proportional to $r^{\lambda-1}$ they would be infinite at the crack tip, but would diverge so fast (faster than $1/r$) that the strain energy would be divergent, so $\lambda < 0$ is not possible.

Finally, if we tried a fraction power, λ, other than $\lambda = ½$, it would not give $\sigma_{yz} = 0$ on the crack faces. This is because $\sigma_{yz} \propto \cos(\lambda\pi)$ on the crack faces, and $\lambda = ½$ is the only fractional λ for which this is zero.

So, $\lambda = ½$ is the only possibility.

Qu.: So what is the message from all that maths?

The message is that the defining essence of a crack is discontinuity of displacements over the crack. So the solutions need a √ to generate this discontinuity, and that's why the stresses end up varying as $1/\sqrt{r}$. It is the discontinuity over the crack that produces the $1/\sqrt{r}$.

In analytic function theory terms, a crack is a branch cut.

> The square-roots which abound in fracture mechanics, and the peculiar unit "MPa√m", arise as a consequence of the crack being a discontinuity in the displacement field and the requirement for zero surface stresses on the crack faces.

Qu.: Can the Mode I and II LEFM Fields be found in a similar manner?

Yes. The algebra is a bit messier because the fundamental equation is the biharmonic equation $\nabla^4 \chi = 0$ where χ is the Airy function, rather than the simpler harmonic equation, $\nabla^2 u_z = 0$. This leads to solutions in terms of two analytic functions rather than just one – which is why the algebra is messier. But the important feature is just the same. To generate the discontinuity over the crack, the functions in question have a √ type branch cut, leading once again to stresses proportional to $1/\sqrt{r}$. The derivation of the Mode I and II crack tip fields can be found on my web site or in standard texts (http://rickbradford.co.uk/DerivationofLEFMFields.pdf)

Qu.: Why is the angular dependence a function of the half-angle, $\theta/2$?

This follows from the same thing – that the complex functions involve square-roots, like $s^{1/2}$, and this gives rise both to the $1/\sqrt{r}$ dependence and also angular dependencies derived from $e^{i\theta/2}$, i.e., functions of the half-angle.

Qu.: So what are the LEFM fields

They are listed below in both polar and Cartesian coordinates. For all modes and all stress components, they are of the form $\sigma \propto \frac{K.f(\theta)}{\sqrt{r}}$, where the function of angle ensures that the crack faces are traction free, and also that the correct stress components are zero on $\theta = 0$, depending upon the Mode.

Qu.: But how is $K_I = \sigma_\infty \sqrt{\pi a}$ derived from elasticity theory?

To derive this we need to introduce the crack length, so we consider an embedded crack of length 2*a* in an infinite plate. We will consider Mode III (anti-plane strain) again, so this is equivalent to an edge crack of length *a* in a semi-infinite plate. The appropriate analytic function for u_z is readily guessed. Since u_z becomes proportional to $\sqrt{s}$ when $|s|$ is small, i.e., near the crack tip, and because one of the crack tips is located at x = a, then $u_z \propto \sqrt{s-a}$ near that crack tip. But there is also a crack tip at x = -a which suggests that $u_z \propto \sqrt{s+a}$ near that crack tip. Overall, therefore, we expect (or guess),

$$u_z = C\,\mathrm{Im}\left(\sqrt{s-a}\cdot\sqrt{s+a}\right) = C\,\mathrm{Im}\left(\sqrt{s^2-a^2}\right)$$

where C is a constant. If this is right then we would get,

$$\frac{\sigma_{xz}}{G} = u_{z,x} = \frac{d}{dx}C\,\mathrm{Im}\left(\sqrt{s^2-a^2}\right) = C\,\mathrm{Im}\frac{d}{dx}\left(\sqrt{s^2-a^2}\right) = C\,\mathrm{Im}\frac{d}{ds}\left(\sqrt{s^2-a^2}\right) = C\,\mathrm{Im}\left(\frac{s}{\sqrt{s^2-a^2}}\right)$$

$$\frac{\sigma_{yz}}{G} = u_{z,y} = \frac{d}{dy}C\,\mathrm{Im}\left(\sqrt{s^2-a^2}\right) = C\,\mathrm{Im}\frac{d}{dy}\left(\sqrt{s^2-a^2}\right) = C\,\mathrm{Im}\,i\frac{d}{ds}\left(\sqrt{s^2-a^2}\right) = C\Re\left(\frac{s}{\sqrt{s^2-a^2}}\right)$$

As $s \to \infty$, the final bracketed term tends simply to 1, and hence has zero imaginary part. Hence, the xz-shear tends to zero at large distances, whilst the yz-shear, which is the Mode III shear component, becomes the constant CG. Hence we can put $CG = \tau_\infty$, where τ_∞ is the remotely applied yz shear stress.

Near the crack tip at x = a we put $s = a + \tilde{s}$, where $\tilde{s} = re^{i\theta}$, and r << a. We can then approximate $\frac{s}{\sqrt{s^2-a^2}} = \frac{a+\tilde{s}}{\sqrt{(s-a)(s+a)}} = \frac{a+\tilde{s}}{\sqrt{(\tilde{s})(\tilde{s}+2a)}} \approx \frac{a}{\sqrt{2a\tilde{s}}} = \sqrt{\frac{a}{2\tilde{s}}} = \sqrt{\frac{a}{2r}}e^{-i\theta/2}$.

Thus, $\sigma_{xz} = \tau_\infty\,\mathrm{Im}\left(\frac{s}{\sqrt{s^2-a^2}}\right) = \tau_\infty\,\mathrm{Im}\left(\sqrt{\frac{a}{2r}}e^{-i\theta/2}\right) = -\tau_\infty\sqrt{\frac{a}{2r}}\sin\left(\frac{\theta}{2}\right) = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right)$,

where we have put $K_{III} = \tau_\infty\sqrt{\pi a}$. In the same way we get $\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)$.

Qu.: But why are we interested in K anyway?

Because in place of a failure criterion based on a critical failure stress, we base the criterion for brittle fracture on reaching a critical value of K, called "fracture toughness" or K_{Ic}.

LEFM Criterion for Brittle Fracture
$K_I = K_{Ic}$ = fracture toughness

Qu.: How does this get around the little problem of the stresses being infinite?

Most obviously, K is finite.

Qu.: Why must fracture be specified by a critical value of K?

...because K defines all the stress and strain components near the crack tip via the unique LEFM fields.

Qu.: But the LEFM fields break down sufficiently near the crack tip....?

> The argument is that it does not matter that the LEFM crack tip fields do not necessarily prevail within the "process zone" within which the complicated fracture processes are taking place. So long as there is a region surrounding the crack tip within which the LEFM fields are a reasonable approximation, then the LEFM fields control what is happening within the process zone. The LEFM fields act as the boundary condition for the process zone. Consequently, K must control fracture since K controls the LEFM fields.

LEFM Crack Tip Fields in Polar Coordinates

Mode I

Eqs.(1)

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[-\cos\frac{3\theta}{2} + 5\cos\frac{\theta}{2}\right] \qquad u_r = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(\frac{5}{2} - 4\bar{\nu}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2} + 4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right]$$

Mode II

Equs.(2)

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} - 5\sin\frac{\theta}{2}\right] \qquad u_r = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{5}{2} + 4\bar{\nu}\right)\sin\frac{\theta}{2} + \frac{3}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_\theta = -\frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2}\right] \qquad u_\theta = \frac{K_{II}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\left[\left(-\frac{7}{2} + 4\bar{\nu}\right)\cos\frac{\theta}{2} + \frac{3}{2}\cos\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \cdot \frac{1}{4}\left[3\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right]$$

In Equs.(1, 2), $\bar{\nu} = \dfrac{\nu}{1+\nu}$ in plane stress, but $\bar{\nu} = \nu$ in plane strain.
In plane stress $\sigma_z = 0$, whereas in plane strain $\sigma_z = \nu(\sigma_r + \sigma_\theta)$.

Mode III

Equs.(3)

$$\sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \qquad \sigma_{\theta z} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \qquad u_z = \frac{4K_{III}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin(\theta/2)$$

LEFM Crack Tip Fields in Cartesian Coordinates

Mode I Equs.(4)

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad u_x = \frac{2(1+\nu)K_I}{E}\sqrt{\frac{r}{2\pi}}\left[1-2\bar{\nu}+\sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad u_y = \frac{2(1+\nu)K_I}{E}\sqrt{\frac{r}{2\pi}}\left[2-2\bar{\nu}-\cos^2\frac{\theta}{2}\right]\sin\frac{\theta}{2}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Mode II Equs.(5)

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \qquad u_x = \frac{2(1+\nu)K_{II}}{E}\sqrt{\frac{r}{2\pi}}\left[2-2\bar{\nu}+\cos^2\frac{\theta}{2}\right]\sin\frac{\theta}{2}$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \qquad u_y = \frac{2(1+\nu)K_{II}}{E}\sqrt{\frac{r}{2\pi}}\left[-1+2\bar{\nu}+\sin^2\frac{\theta}{2}\right]\cos\frac{\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

In Equs.(19, 20), $\bar{\nu} = \frac{\nu}{1+\nu}$ in plane stress, but $\bar{\nu} = \nu$ in plane strain.
In plane stress $\sigma_z = 0$, whereas in plane strain $\sigma_z = \nu(\sigma_{xx} + \sigma_{yy})$.

Mode III Equs.(6)

$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right) \qquad \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \qquad u_z = \frac{4K_{III}(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin(\theta/2)$$