Tutorial Session 9 T72S01

Relates to T72S01 Knowledge & Skills 1.14, 1.15, 1.16, 1.17, 1.18

Last Update: 4/5/14

The lower bound theorem of plastic collapse: proof and example applications; The reference stress concept: definition and examples; cases with more than one applied load.; concept of primary and secondary loads and its limitations; elastic follow-up.

Qu.: In general, how is the shape of the yield surface constrained?

The yield surface in deviatoric stress space is a convex closed loop.

Qu.: What is the direction of plastic flow?

The vector describing the increment of plastic strain in deviatoric stress space is normal to the yield surface.

Qu.: What do normality and convexity imply for the strength of structures?

We shall see that it is the **convexity** of the yield surface, and the fact that the plastic strain increment is **normal** to the yield surface that results in the Upper and Lower Bound Theorems.

Qu.: What is the Lower Bound Theorem of plasticity?

If some postulated distribution of stresses within a body is;
(a) in equilibrium everywhere, and,
(b) in equilibrium with some applied loads, P_i, and,
(c) the equivalent stress does not exceed the yield stress anywhere,
then the loads P_i are a lower bound estimate of the loads required to cause plastic collapse.

Qu.: What use is the lower bound theorem?

It is the basis of most plastic collapse load estimates used in structural analysis. Hence, it is the basis of most references stresses used in both R5 and R6 assessments, and thus underpins almost everything that we do. It is the most important theorem in structural analysis.

Qu.: What role do displacements and strains play in the lower bound theorem?

None.

The displacements and strains are irrelevant and play no part in the lower bound theorem.

Qu.: What exactly does being "in equilibrium everywhere" mean?

Being "in equilibrium everywhere" means that the stress distribution obeys $\frac{\partial \sigma_{ij}}{\partial x_j} = b_i$ at every point in the body (where b_i is the body force per unit volume, most often zero). It also requires that the stresses are in equilibrium with the applied loads at the points where these are applied, e.g., this might mean certain components of σ_{ij} are specified on some boundaries.

Qu.: What yield criterion is required for the lower bound theorem to be true?

Different equivalent stresses can be chosen according to which theory of plasticity one favours. More generally the condition that "the equivalent stress does not exceed yield anywhere" can be replaced by "the yield condition is not violated anywhere", which is taken to be synonymous with "being on or within the yield surface".

Qu.: What types of applied load does the lower bound theorem relate to?

The loads P_i may be any combination of point loads, pressures, tractions, body forces etc, as long as they are all *load controlled* loadings.

Qu.: What if the guessed stress distribution is a really bad guess?

For a poor guess regarding the stresses, the loads P_i may be a uselessly low lower bound – but always safe.

Qu.: What corollaries follow from the lower bound theorem?

One corollary is that you cannot reduce the plastic collapse load of a structure by adding material to it. This follows immediately because a possible stress distribution within yield is obtained from the actual stress distribution of the original body by taking the additional material to be stress-free. The collapse load of the original body is thus a lower bound to that of the enhanced body.

Qu.: But that's just trivial, isn't it?

No.

The fact that it isn't trivial is emphasised by the fact that it is ***untrue*** for the failure loads of real structures in which fracture effects are significant (see "limitations", below). For example, we could add some material which contained a crack and hence give rise to fracture.

A further example is thermal fatigue. Making a section thicker will increase thermal transient stresses – and hence lead to a greater propensity to generate fatigue cracks.

Yet another example is reheat cracking. Making a section thicker increases the constraint (triaxiality) leading to a greater likelihood of reheat cracks forming.

All these examples of cases where *adding* material is structural deleterious relate to mechanisms other than plastic collapse, being controlled by toughness, ductility, creep ductility or fatigue endurance (tolerance to cyclic strains).

Qu.: What about removing material?

The lower bound theorem also implies that removing material cannot increase the plastic collapse load. To see this recall that any material to be removed can instead be assigned zero stress. This merely limits the range of options for stress distributions and hence can only reduce (or leave unchanged) the candidate collapse load.

Qu.: Is this also untrue for the failure loads of real structures?

Yes.

For example, a common means of ameliorating the effects of a crack is to drill a 'crack stopper' hole around the crack tip. This removal of material blunts the crack tip and hence increases its fracture load.

Reducing a section thickness will make it less susceptible to thermal stresses and reheat cracking – though primary stresses will be increased.

Qu.: What are the limitations of applicability of the lower bound theorem?

The Lower Bound Theorem applies rigorously for,

a) The plastic collapse load, not to other failure mechanisms;
b) Arbitrarily large plastic ductility is implicit;
c) Elastic-perfectly plastic behaviour;
d) Small strains (body geometry virtually unchanged by deformation);

Qu.: What is the significance of limitations (a) and (b)?

We have already seen that some failure/cracking mechanisms (e.g., fracture, fatigue, creep crack initiation) behave very differently from plastic collapse. This is because fracture or crack initiation occurs due to either, (a) the material's ability to sustain only a limited amount of strain (low plastic ductility or low creep ductility), or, (b) the material's reduced resistance to cracking in the presence of elevated hydrostatic stress, or, (c) cyclic effects repeatedly dumping more energy (damage) into the material.

The yielding which is inherent in redistributing stresses and which, physically, explains to the lower bound theorem for plastic collapse, does not help with issues of limited strain tolerance or limited hydrostatic stress tolerance or limited ability to absorb energy (damage).

Note that hydrostatic stresses do not cause yielding and hence do not lead to their own relaxation. This is why, in R6, there is both a fracture axis and a plastic collapse axis. The fracture failure mode must be addressed separately. Plastic collapse effectively means the failure mode which is not influenced by fracture toughness or finite ductility or other forms of damage.

So (a) and (b) are not really limitations so much as definitions of what is meant by 'plastic collapse'. Plastic collapse is that mechanism which results in immediate failure due to the inability of the structure to arrange its stresses to equilibrate the applied loads.

Qu.: What is the significance of limitation (c)?

This is the most significant limitation as regards the numerical accuracy of applications of the theorem. The reason is simply that there is no uniquely defined 'yield stress'. Real structural steels exhibit large amounts of strain hardening. Specific assessment procedures will advise on the material property to employ in lieu of a perfectly plastic yield stress. For R6 this is the lower bound 0.2% proof stress, $\sigma_{0.2}$ (used to define Lr). However, note the crucial distinction between Lr =1, which means general yield, and actual plastic collapse. A structure does not necessarily fail an R6 assessment if Lr=1.

R6 also uses a "flow" stress, σ_f, to define a better estimate of real plastic collapse. It is defined as the lower bound average of the 0.2% proof stress and the UTS. A maximum permitted value of Lr (the cut-off) is defined as $\sigma_f / \sigma_{0.2}$. However, for many assessments the Lr cut-off plays no part since the failure assessment diagram (FAD) is reached at smaller Lr.

Actually, the apparently arbitrary use of the 0.2% proof stress has less effect on the accuracy of R6 assessments than might have been supposed. The reason is that the shape of the R6 FAD also depends upon the definition of proof strength adopted. So adopting of a different convention would be partly self-compensating via a changed FAD.

Qu.: What is the significance of limitation (d)?

The restriction to small strains is not normally a problem since typically the strains will be modest and the deformation will generally be small. However, there are cases when the assumption can be non-conservative, e.g. an upward slanted cantilever for which the maximum bending moment increases as the structure deforms at constant load.

There are also cases where it can be grossly over-conservative to assume small displacements, e.g. an initially out-of-round internally pressurised pipe. In cases like this, however, the offending wall-bending stresses are really secondary, and recognising this obviates the problem.

Good practice when using FEA for collapse is to obtain results both with and without updated geometry. R6 advises the use of small displacement theory (NLGEOM OFF in ABAQUS speak). However, my opinion is that there is no general rule as to which of small or large displacements is the more onerous or the more accurate, and I advise always using the more onerous of the two.

Qu.: What is the proof of the lower bound theorem?

- Recall that we are assuming some distribution of stresses, σ_{ij}, which are everywhere in equilibrium and in equilibrium with some applied loads, P_i, and such that the equivalent stress does not exceed the yield stress anywhere.
- The principal deviatoric stresses corresponding to the postulated stress distribution, σ_{ij}, are denoted σ_K, where the subscript 'K' refers to the 2 directions of principal deviatoric space. The deviatoric stress σ_K is on or within the yield surface.
- Suppose that scaling all the applied loads, P_i, by some factor, λ, just results in collapse. Then the correct collapse loads are $\tilde{P}_i = \lambda P_i$. We wish to demonstrate that $\lambda \geq 1$ so that our applied load is a lower bound to the correct collapse load.
- Denote the correct plastic strain increments as collapse is approached by $d\tilde{\varepsilon}_K^p$, where the subscript 'K' again refers to the 2 directions of principal deviatoric space. The plastic strain increment is effectively a 2D vector as shown in the diagram below.
- Call the displacement increments as collapse is approached $d\tilde{u}_i$, where 'i' denotes the usual Cartesian directions in 3D space. Hence $d\tilde{u}_i$ are kinematically (i.e., geometrically) compatible with the strains $d\tilde{\varepsilon}_K^p$. They must be because they are both the correct values.
- The correct stress distribution at collapse is written $\tilde{\sigma}_K$, where again 'K' denotes the 2 directions of principal deviatoric space. Hence, $\tilde{\sigma}_K$ are in equilibrium with the correct collapse loads, $\tilde{P}_i$.
- The principle of virtual work states that the work done by the hypothetical loads P_i must equal the integral of the work done on each element of the body by the hypothetical stress field with which it is in equilibrium, i.e.,

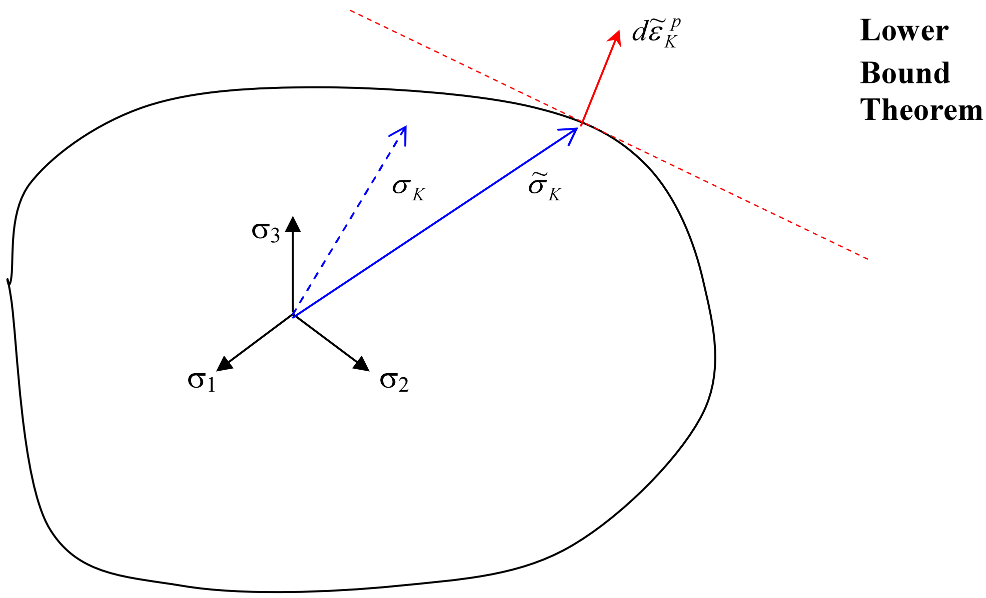
$$\sum_i P_i d\tilde{u}_i = \int \sigma_K d\tilde{\varepsilon}_K^p dV \qquad (1)$$

The repeated indices in (1) are summed (i.e., dot products).

The principle of virtual work applies because σ_K and P_i are in equilibrium and $d\tilde{\varepsilon}_K^p$ and $d\tilde{u}_i$ are kinematically compatible. Of course, the same relationship holds also for the correct collapse load and stresses, i.e.,

$$\sum_i \tilde{P}_i d\tilde{u}_i = \int \tilde{\sigma}_K d\tilde{\varepsilon}_K^p dV \qquad (2)$$

Wherever the plastic strain increment is non-zero, the actual stress $\tilde{\sigma}$ must lie on the yield surface. On the other hand, the hypothetical stress σ lies on or within the yield surface, by hypothesis. Hence, in the 2D deviatoric stress plane we have,



It is clear from the above diagram that,

$$\sigma_K d\tilde{\varepsilon}_K^p \le \tilde{\sigma}_K d\tilde{\varepsilon}_K^p \qquad (3)$$

at all points in the body. In (3) the repeated indices indicate the dot product. From (1), (2) and (3) it therefore follows that,

$$\sum_i P_i d\tilde{u}_i \le \sum_i \tilde{P}_i d\tilde{u}_i = \sum_i \lambda P_i d\tilde{u}_i = \lambda \sum_i P_i d\tilde{u}_i \qquad (4)$$

and hence that, $\lambda \ge 1$

Thus, our postulated load is a lower bound to the correct collapse load. QED.

Qu.: Why does the elastic energy play no part in this proof?

The alert reader will have noticed that the proof assumes that the increments of elastic strain, $d\varepsilon_{ij}^{el}$, are zero at collapse, and hence that the elastic strain energy increment is zero as collapse is approached. That is, that the elastic regions behave rigidly at collapse. This is where the assumption of perfect plasticity, i.e. no hardening, is important. In perfect plasticity, the stresses will be unchanging as plastic flow proceeds at close to collapse. Since the stresses are unchanging, the elastic strains are also unchanging, as assumed. But for a material which is still hardening even as collapse is approached, the increasing elastic strain energy would screw-up the proof: hence the assumption of perfect plasticity.

Qu.: What does the lower bound theorem mean physically?

As the load is increased, plasticity spreads through the structure causing more and more redistribution of the stresses until the only thing that can bring the process to an end is the inability to satisfy both equilibrium and the yield condition.

Challenge: Probably, at the limit load, the equivalent stress reaches the yield stress along the whole of a continuous surface which divides the structure into two parts…*but I've never seen this proved.*

Qu.: What is the collapse load for a Rectangular Section Bar In Bending?

The bar width is B and its thickness is t. The maximum magnitude of axial stress consistent with the yield condition is σ_y (because the stressing is uniaxial). Assume $+\sigma_y$ on one side of the neutral axis and $-\sigma_y$ on the other side. The load carried by one half of the section is thus $\sigma_y Bt/2$, and the centre of this load is at a distance of t/4 from the centre of the bar. It therefore contributes a moment of $\sigma_y Bt^2/8$. The other half contributes an equal moment, making $\sigma_y Bt^2/4$ in all. This results in a lower bound estimate of the collapse moment of,

$$M_y = \frac{Bt^2\sigma_y}{4} = \frac{3}{2} M_{yield}^{first} \qquad (5)$$

where the "first yield" moment relates to the onset of yielding on the outer fibre, i.e. the maximum elastic stress reaching yield. As always, the characteristic stress that is used as "σ_y" is debatable.

The ratio of the collapse moment to the moment to cause first yield will always be greater than 1. For the rectangular section bar, Equ.(5) shows that it is 3/2. This is the origin of the factor of 1.5 by which design codes often permit membrane-plus-bending stresses to exceed the membrane design limit. However, for global bending of other sections this ratio will generally be different.

Qu.: What is the collapse load of a Thin Cylinder under Global Bending?

The axial stress is again assumed to be $+\sigma_y$ on one side of the neutral axis and $-\sigma_y$ on the other side. If y is the distance from the neutral axis the moment contributed by one quarter of the shell is $\int_0^{\pi/2} t\sigma_y y ds = \int_0^{\pi/2} t\sigma_y r\sin\theta . r d\theta = t\sigma_y r^2$. So the total 'collapse' moment is $M_y = 4t\sigma_y r^2$. For a thin shell the section modulus is $I = \pi t r^3$ and so the moment at first yield of the outer fibre is given by $M_{yield}^{first} = I\sigma_y / r = \pi t\sigma_y r^2$. Hence, $M_y = \frac{4}{\pi} M_{yield}^{first}$. The ratio in this case is only $4/\pi$, significantly less than 1.5.

Qu.: What is the collapse load of a Thick Pipe With Internal Pressure?

The equilibrium equation in polar coordinates is,

$$r\frac{\partial \sigma_r}{\partial r} + (\sigma_r - \sigma_h) = 0 \qquad (6)$$

where the subscripts denote the radial and hoop stresses. Assuming the yield stress is reached everywhere (the limit condition) gives $\sigma_h - \sigma_r = \sigma_y$ assuming Tresca yield theory (and noting that the radial stress is compressive and the hoop stress tensile). Hence, Equ.(6) becomes simply,

$$r\frac{\partial \sigma_r}{\partial r} = \sigma_y \qquad (7)$$

The boundary conditions are that the radial stress is $-P_y$ (the collapse pressure) on the inner radius (r = a) and zero on the outer radius (r = b). Hence, integrating gives,

$$\int_a^b d\sigma_r = \sigma_r(b) - \sigma_r(a) = P_y = \int_a^b \frac{\sigma_y}{r} dr = \sigma_y \log\left(\frac{b}{a}\right) \qquad \text{(Tresca)} \qquad (8)$$

which is the usual 'log' solution for a thick cylinder. (log is natural).

Note that *any* stress distribution which respects equilibrium results in this conclusion, and hence (8) is the maximum lower bound, i.e., in this case we have actually found the correct (Tresca) solution.

Although we have not derived the Mises solution directly here, this problem is plane strain – hence we already know that the Mises collapse pressure will simply be a factor of $2/\sqrt{3}$ larger than (8).

Qu.: Is P_y the collapse load or the general yield load?

That depends upon what material property, σ_y, has been used in its definition. If this is the 0.2% proof stress, then P_y is really the yield load (and, more precisely, the yield load at the 0.2% level). But if some hardened stress is used, e.g. the flow stress, then P_y might loosely be called the plastic collapse load. In truth it is probably most accurate to use the UTS here, to get the best estimate of the collapse load. But, for conservatism, codes like R6 recommend a flow stress defined as the average of the 0.2% proof stress and the UTS. Use of the UTS in an assessment of collapse would implicitly permit huge distortions prior to the calculated collapse load.

Qu.: What is a 'Reference Stress'?

The reference stress is defined (e.g., in R6) as,

$$\sigma_{ref} = \frac{P}{P_y}\sigma_y \qquad (9)$$

where P is the applied load and P_y is the general yield (collapse) load, and σ_y is the proof stress used to define P_y.

Qu.: Does the reference stress depend on σ_y ?

No.

The proof stress, σ_y, cancels between numerator and denominator in (9), so the reference stress does not depend upon the material strength at all. It depends only upon the geometry of the body and the nature and magnitude of the loading.

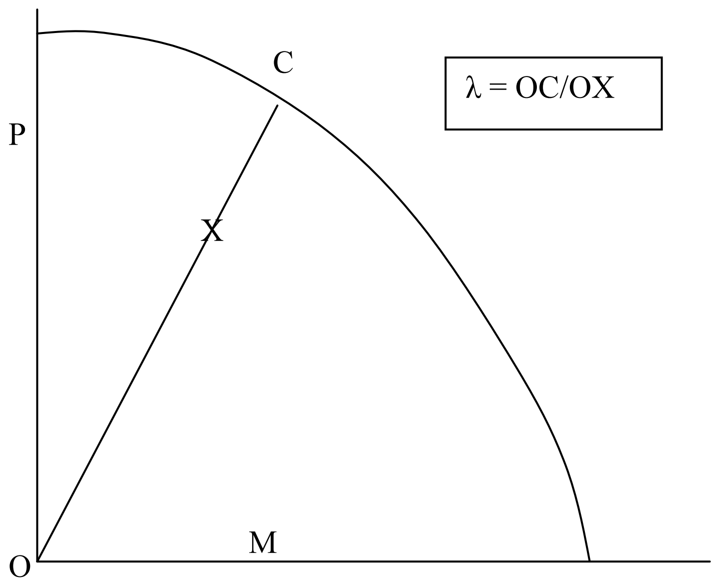
Very roughly, the reference stress can be thought of as a sort of average of the equivalent stress over the section which would collapse.

Qu.: What if there are two or more loads acting?

As far as I know neither R6 nor R5 are explicit about the definition of reference stress when two or more loads are acting. However, I believe the following is implicit.

Suppose there are two loads, P and M, acting. One first finds the factor, λ, by which ***both*** loads must be increased to result in general yielding (collapse). It is important that both loads are increased in proportion. The reference stress is then defined by

$$\sigma_{ref} = \sigma_y / \lambda \qquad (10)$$



Qu.: What is a primary load?

A primary load does not relax when yielding occurs. It corresponds to a direct application of forces, moments, weights, pressures, etc.

Qu: What is a secondary load?

It is tempting to say that secondary loads are loads which are not primary. But in practice the term ‘secondary load’ is used for only a subset of non-primary loads. Secondary loads are exemplified by applied strains, or displacements applied over a short gauge length. Their characteristic is that they are readily relaxed by the accumulation of a modest amount of irrecoverable strain (i.e., plastic or creep strain).

Secondary loads are loads which do not contribute to plastic collapse. They relax to arbitrarily small levels if sufficient yielding takes place.

Welding residual stresses are generally secondary. Thermal stresses are often secondary, but not always. Applied displacements may or may not be secondary.

Qu.: What loads are neither primary nor secondary?

In many cases, loads which might potentially be secondary (thermal stresses, applied displacements) cannot safety be assumed to be secondary. This is because the plastic strain required to relax them may be larger than would accumulate prior to failure. Such loads may be termed ‘displacement controlled’, or ‘secondary-with-follow-up’.

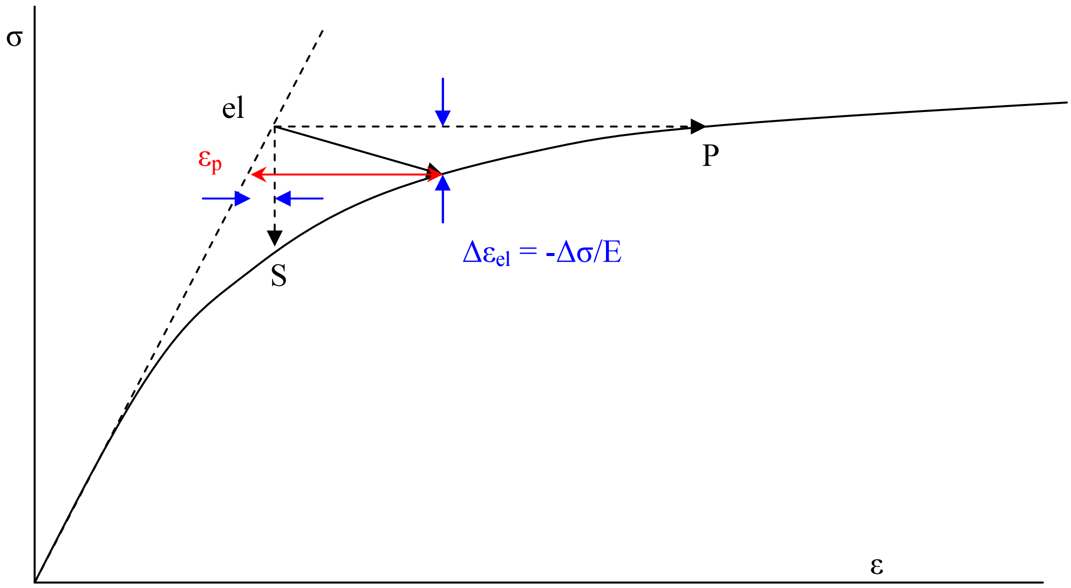
Qu.: What is elastic follow-up?

Elastic follow-up is the mechanism whereby the predominantly elastic nature of the remote material throws load onto a yielding region and prevents it from relaxing as much as would otherwise have been expected.

Qu.: How is the elastic follow-up factor, Z, defined?

With reference to the stress-strain diagram below: suppose a point is loaded so that, if it remained elastic, it would be at the point labelled “el”. The curve in the diagram is the stress-strain hardening curve of the material. Hence the point “el” is fictitious since the stress-strain point must lie on this curve. For a purely primary stress, the stress is fixed during yielding (at least if we assume all redistribution has been completed and the structure is nearing collapse). Hence, a primary stress would actually be at the point labelled P. In contrast, a secondary stress corresponds to a fixed total strain, and hence yielding takes us to the point S. In reality, loads and structures will generally behave in an intermediate manner, with the stress-strain point being at some intermediate position, as indicated. This corresponds to a partial relaxation of the stress by an amount $\Delta\sigma$, and hence a reduction in the elastic strain by an amount $\Delta\varepsilon_{el} = -\Delta\sigma / E$. But it also corresponds to an accumulation of plastic strain by some amount ε_p, whose graphical interpretation is indicated. The elastic follow-up factor is defined as,

$$Z = \frac{\varepsilon_p}{|\Delta\varepsilon_{el}|} \tag{11}$$



Hence, for a purely secondary load Z = 1, whereas for a primary load $Z \to \infty$.

Qu.: What is the Convexity Theorem?

The Convexity Theorem is the statement that, if several loads, P_i, are applied to a structure, then the locus in Pi space at which collapse (general yield) occurs is convex. This is actually a corollary of the lower bound theorem. The proof is as follows. Suppose we have two load vectors, $\overline{P}$ and $\overline{Q}$, both of which are collapse loads, i.e., both lie on the collapse surface. Suppose load P is in equilibrium with stress distribution σ^P, and load Q is in equilibrium with stress distribution σ^Q. Consider the stress distribution formed as a linear superposition $\sigma = x\sigma^P + (1-x)\sigma^Q$. Because the equilibrium equation is linear, $\sigma_{ij,j} = b_i$, this stress distribution is in equilibrium everywhere and corresponds to applied load vector $\overline{F} = x\overline{P} + (1-x)\overline{Q}$. Moreover, because the yield surface in stress space is convex, and because both σ^P and σ^Q lie on or within the yield surface, it follows that σ must lie on or within the yield surface. Consequently, since σ dos not violate the yield condition and is in equilibrium everywhere, it follows that $\overline{F} = x\overline{P} + (1-x)\overline{Q}$ is on or within the yield surface in load-space. But this applies for arbitrary x and arbitrary collapse loads $\overline{P}$ and $\overline{Q}$, so this defines a convex surface. QED.