**Tutorial Session 8c: T72S01**
**Basic Plasticity: Part 3 – The Deviatoric Stress Plane**
Relates to T72S01 Knowledge & Skills 1.2, 1.3
*The deviatoric stress plane; the Mises circle and the Tresca hexagon*
Last Update: 26/2/14

Qu.: What are the deviatoric stresses?

The deviatoric stresses are the on-diagonal (direct) stresses minus the hydrostatic stress. They are denote by a 'hat' (caret): thus, $\hat{\sigma}_x = \sigma_x - \sigma_H$. The shear stresses are not affected. Hence,

$$\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_H \delta_{ij} \qquad \text{or} \qquad \hat{\sigma} = \sigma - \sigma_H \mathrm{I} \tag{1}$$

where $\mathrm{I}$ is the unit matrix and the Kronecker symbol $\delta_{ij}$ are its components, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Qu.: Why do the deviatoric stresses matter?

For isotropic materials, the hydrostatic stress component does not contribute to yielding (i.e. the equivalent stresses do not change if the hydrostatic stress alone is changed). Hence, yield criteria are expressed in terms of the stresses which remain after the hydrostatic stress is subtracted off, i.e. the deviatoric stresses.

Qu.: Why does the yield of isotropic media not depend upon the hydrostatic stress?

This is a consequence of the incompressibility of plastic deformations (which in turn results from the fact that plastic flow relates to the sliding of planes of atoms, i.e. dislocation glide). The volumetric strain is three times the hydrostatic strain, i.e., $\varepsilon_x + \varepsilon_y + \varepsilon_z$, and this is zero for plastic deformations, $\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p = 0$. But if a hydrostatic stress were to cause plastic strain, it could only cause hydrostatic straining in an isotropic medium, i.e., it would result in a non-zero $\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p$. So hydrostatic stress cannot cause yielding.

Qu.: Why is the volumetric strain equal to $\varepsilon_x + \varepsilon_y + \varepsilon_z$ (to first order)

The deformed length of a sample of material in the x-direction is a factor $1 + \varepsilon_x$ times the undeformed length, by definition of strain. Hence, the deformed volume is a factor $(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$ times the original volume. Expanding and retaining only the term of leading (linear) order gives the change of volume as $\delta V = (\varepsilon_x + \varepsilon_y + \varepsilon_z)V_0$. Hence the volumetric strain is,

$$\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{2}$$

Qu.: What is the hydrostatic strain?

My definition is that the hydrostatic strain is, in analogy to the hydrostatic stress, the average of the three direct strains – and hence equal to one-third of the volumetric strain,

$$\varepsilon_H = \frac{1}{3}\varepsilon_V = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{3}$$

Qu.: What are the deviatoric strains?

The deviatoric strains are defined just like the deviatoric stresses, via,

$$\hat{\varepsilon}_{ij} = \varepsilon_{ij} - \varepsilon_H \delta_{ij} \qquad \text{or} \qquad \hat{\varepsilon} = \varepsilon - \varepsilon_H \mathrm{I} \tag{4}$$

The usual 3D isotropic Hooke's Law for linear elastic media gives $E\varepsilon_H = (1-2\nu)\sigma_H$.

Qu.: What is the value of Poisson's ratio in plasticity?

Strictly speaking there is no such thing as Poisson's ratio in plasticity, since there is no linear relationship between stress and strain to define a constant of proportionality. However, if we define $\nu^p = -\varepsilon_y^p / \varepsilon_x^p$ in a situation where $\varepsilon_y^p = \varepsilon_z^p$ then the fact that $\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p = 0$ gives $\nu^p = ½$. This value for Poisson's ratio is indicative of incompressibility.

Qu.: What is the Mises yield criterion?

The Mises yield criterion is that $\bar{\sigma} = \sigma_0$, where $\sigma_0$ is the yield strength.

Qu.: What is the Mises equivalent stress?

In terms of the deviatoric stresses, the Mises stress is defined by $\bar{\sigma} = \sqrt{\frac{3}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij}}$, recalling that repeated indices are summed. In terms of principal deviatoric stresses the Mises criterion is thus $\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2 = 2\bar{\sigma}^2/3$. This reduces to the usual definition in arbitrary coordinates,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)\right]^{1/2} \tag{5}$$

Qu.: What is the Tresca stress and the Tresca yield criterion?

The Tresca stress is the difference between the algebraically largest and smallest of the principal stresses,

$$\sigma_T = \max(\sigma_1, \sigma_2, \sigma_3) - \min(\sigma_1, \sigma_2, \sigma_3) \tag{6}$$

> NB: In the notes for this session we differ from the usual convention that $(\sigma_1, \sigma_2, \sigma_3)$ are in order of size – instead they can be in any order in this session.

> NB: In these notes we assume an isotropic material throughout

Qu.: What shape is the yield surface?

In the previous version of this session I proceeded directly to the deviatoric stress plane. This is because the yield surface is ***far*** simpler in the deviatoric stress plane. The pedagogic disadvantage of this is that people failed to appreciate just how much simpler it is to work in the deviatoric stress plane. To really appreciate this it is necessary first to do things the hard way…(and text books still do!)…

Qu.: So - what shape is the yield surface?

Of course I could make things really nasty by working in an arbitrary coordinate system – in which case there are 6 stress components and hence the yield surface lives in a 6-dimensional space. We'll not go that far.

Instead we shall work throughout in principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. Consider the Mises yield criterion,

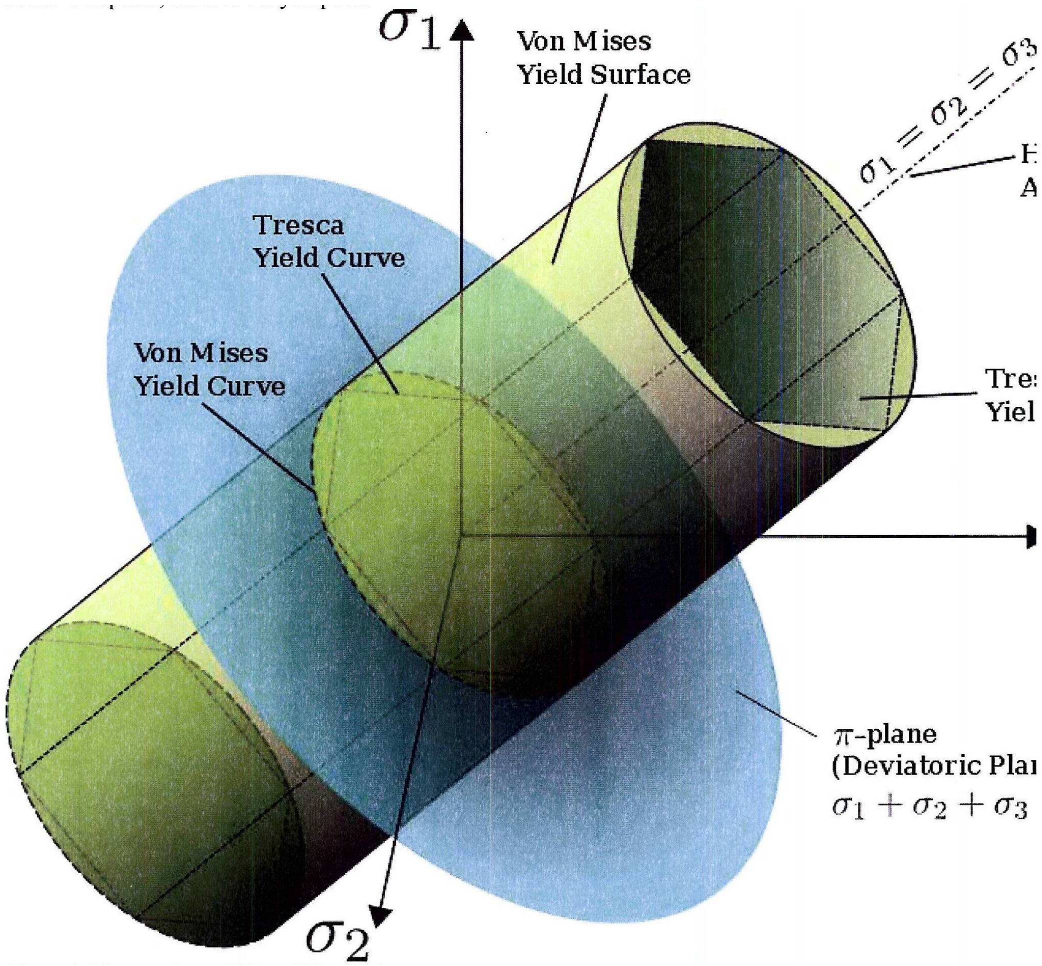
$$\frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sigma_0 \quad (7)$$

where $\sigma_0$ is some appropriate yield stress. This defines a surface in the 3D stress space defined with axes $(\sigma_1, \sigma_2, \sigma_3)$.

Qu.: So - what shape ***is*** the Mises yield surface given by Equ.(7)?

It's not immediately obvious, is it? But this is rather my point. If you think about it enough you'll discover that (7) defines the surface of a cylinder of radius $\sigma_0$ and whose axis lies along the line through the origin and in the $(1,1,1)$ direction in $(\sigma_1, \sigma_2, \sigma_3)$ space. So it looks like this (courtesy of Prof.Wiki)…

**Figure 1: Yield Surfaces in 3D Principal Stress Space**




Similarly the Tresca yield criterion gives the hexagonal prism, inscribed within the Mises cylinder. Park this for a moment and let's look at some 2D cases…

Qu. Case 1: What are the yield curves in 2D plane stress?

Suppose $\sigma_2 = 0$, so that $(\sigma_1, \sigma_3)$ is a 2D state of plane stress. The Mises criterion, (7), becomes,

$$\frac{1}{\sqrt{2}}\left[(\sigma_1)^2 + (\sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sigma_0$$

which simplifies to, $$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = \sigma_0^2 \qquad (8)$$

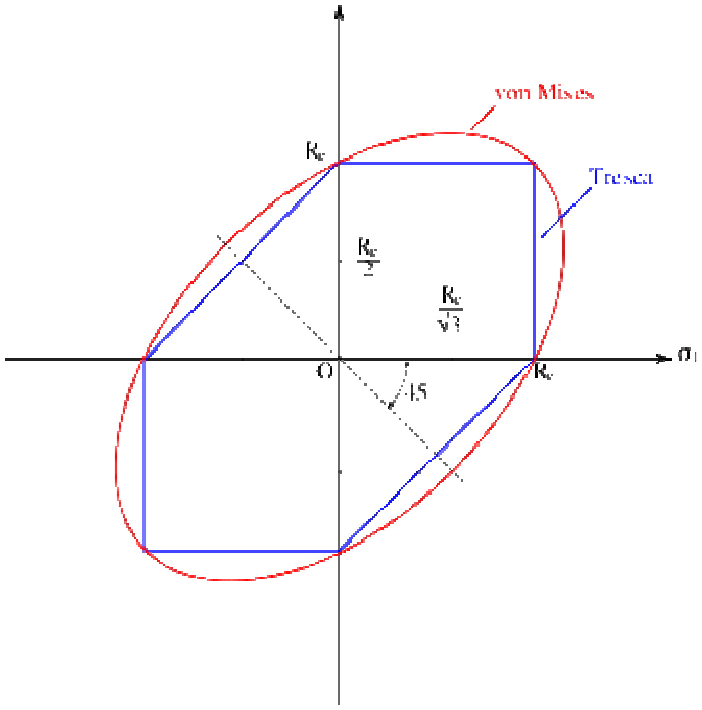
This is the equation of an ellipse whose semi-major axis is in the direction $(1,1)$ in the $(\sigma_1, \sigma_3)$ plane, and whose semi-minor axis is in the direction $(-1,1)$. This can be seen be re-arranging (8) as,

$$\left(\frac{\sigma_1 + \sigma_3}{2\sigma_0}\right)^2 + \left(\frac{\sigma_1 - \sigma_3}{2\sigma_0 / \sqrt{3}}\right)^2 = 1 \qquad (9)$$

which is indeed the equation of an ellipse oriented at $45^o$ to the $\sigma_1$ axis.

**Figure 2: Plane Stress Yield Curves in the $(\sigma_1, \sigma_3)$ Principal Stress Plane, $\sigma_2 = 0$**

(here $R_e$ is the yield stress, $\sigma_0$)




The Tresca yield curve is the inscribed, elongated, hexagon, because…

- Where $(\sigma_1, \sigma_3)$ have the same sign then $\sigma_T = \max(\sigma_1, \sigma_3)$
- Where $(\sigma_1, \sigma_3)$ have the opposite sign then $\sigma_T = |\sigma_1 - \sigma_3|$

Qu.: How does Figure 2 relate to Figure 1?

Since plane stress has $\sigma_2 = 0$, Figure 2 is just the cross-section of Figure 1 defined by the plane $\sigma_2 = 0$.

Qu.: Figure 2 seems simple enough, so why bother with the deviatoric plane?

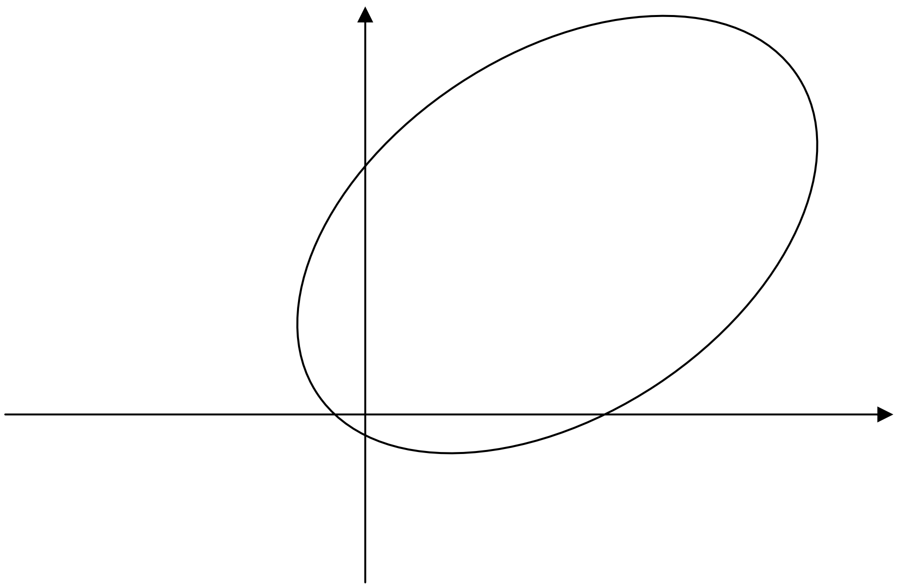
Simply that,

- An ellipse is not as simple as a circle (see below),
- ..and Figure 2 only applies in plane stress!!

Qu. Case 2: What are the yield curves if $\sigma_2$ is constant but non-zero?

In this case we just take the section of Figure 1 defined by $\sigma_2 = A$ (some constant) and we get something like,

**Figure 3: Mises Yield Curve in the $(\sigma_1, \sigma_3)$ Principal Stress Plane for $\sigma_2 = A$**



Convinced yet that this is getting more awkward to deal with?

Qu. Case 3: What are the yield curves in plane strain?

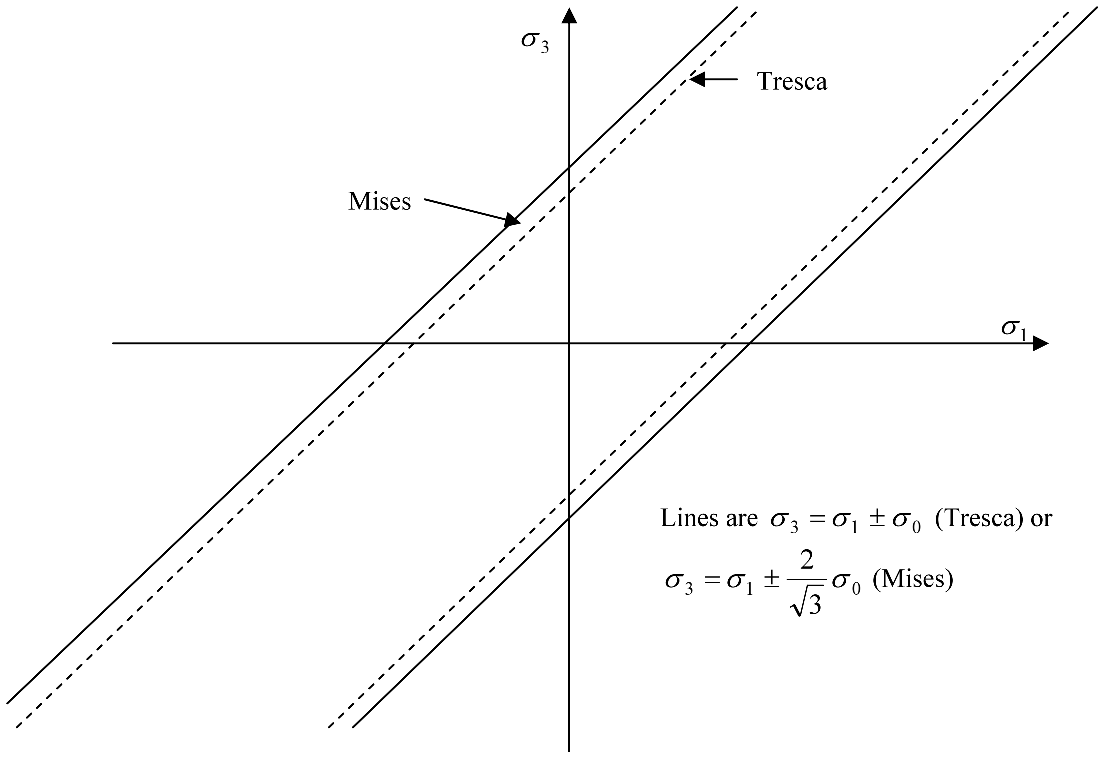
Adherents of working in principal stress space, rather than deviatoric stress space, don't generally plot this one. Let's work it out. If $(\sigma_1, \sigma_3)$ is 2D plain strain, i.e., $\varepsilon_2 = 0$, then we have $\sigma_2 = (\sigma_1 + \sigma_3)/2$ (we'll see why later). Inserting this into the Mises criterion, (7), gives,

$$\frac{1}{\sqrt{2}}\left[\left(\sigma_1 - \frac{\sigma_1+\sigma_3}{2}\right)^2 + \left(\frac{\sigma_1+\sigma_3}{2} - \sigma_3\right)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} =$$
$$\frac{1}{\sqrt{2}}\left[\left(\frac{\sigma_1-\sigma_3}{2}\right)^2 + \left(\frac{\sigma_1-\sigma_3}{2}\right)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \frac{\sqrt{3}}{2}\left|\sigma_1 - \sigma_3\right| = \sigma_0 \quad (10)$$

Hence, the Mises criterion is the same as the Tresca criterion in plane strain except with an elevated effective yield stress of $2\sigma_0/\sqrt{3}$ (which we have seen previously).

But what do the yield 'curves' look like? They are just a pair of parallel tram-lines…

**Figure 4: Plane Strain Yield 'Curves' – they are not closed!**




I can't recall seeing this in the text books…?

Qu.: How does Figure 4 relate to Figure 1?

We need to consider the section of Figure 1 defined by the plane $\sigma_2 = (\sigma_1 + \sigma_3)/2$. But the axis of the cylinder defined by $\sigma_1 = \sigma_2 = \sigma_3$ clearly lies on this plane, because $1 = (1+1)/2$, so this is a section down the axis. So the parallel tram-lines in Figure 4 are just what you get by cutting a cylinder (or hexagonal prism) in half along its length.

| |
|---|
| The purpose of all this has principally been to show how much easier it is to deal with the yield surface in the deviatoric plane….next… |

Qu.: What is the deviatoric plane?

Although there are three deviatoric principal stresses, $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$, it makes no sense to regard this as a 3D space because of the algebraic identity $\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3 = 0$. Consequently, deviatoric stress space is automatically 2D – for any state of stress.

| The deviatoric stress plane is the plane normal to the axis of the cylinder in Figure 1 and passing through the origin (shown blue) |
|---|

Qu.: What shape is the Mises yield surface in the deviatoric plane?

The Mises yield surface, (7), can also be written,

$$\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2 = 2\bar{\sigma}^2/3 = 2\sigma_0^2/3 \qquad (11)$$

This would be a sphere in 3D space, $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$, of radius $\sqrt{2/3} \cdot \sigma_0$, so in the deviatoric plane the yield curve is a section of this sphere, i.e., a circle.

Qu.: What is the Mises yield surface on the deviatoric plane?

A plane intersects a sphere in a circle, so the Mises yield surface is a circle on the deviatoric plane. Since the plane $\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3 = 0$ passes through the origin, the circle has the same radius as the sphere, i.e., $\sqrt{2/3} \cdot \sigma_0$ at yield. However, the deviatoric stress axes do not lie in this plane, but at an angle $\cos^{-1}(1/\sqrt{3})$ to its normal. Because of this, the deviatoric stresses projected onto this plane are a factor of $\sin\cos^{-1}(1/\sqrt{3}) = \sqrt{2/3}$ smaller than the actual deviatoric stresses. But it is convenient to continue to use the deviatoric stresses themselves, rather than constantly scaling everything by $\times\sqrt{2/3}$. This is achieved by also shrinking the yield circle by a factor of $\times\sqrt{2/3}$, so that it now has a radius of $(2/3)\sigma_0$ and we can continue to use the deviatoric stresses as if they lay in this plane.

Qu.: What point on the Mises deviatoric circle corresponds to uniaxial yield?

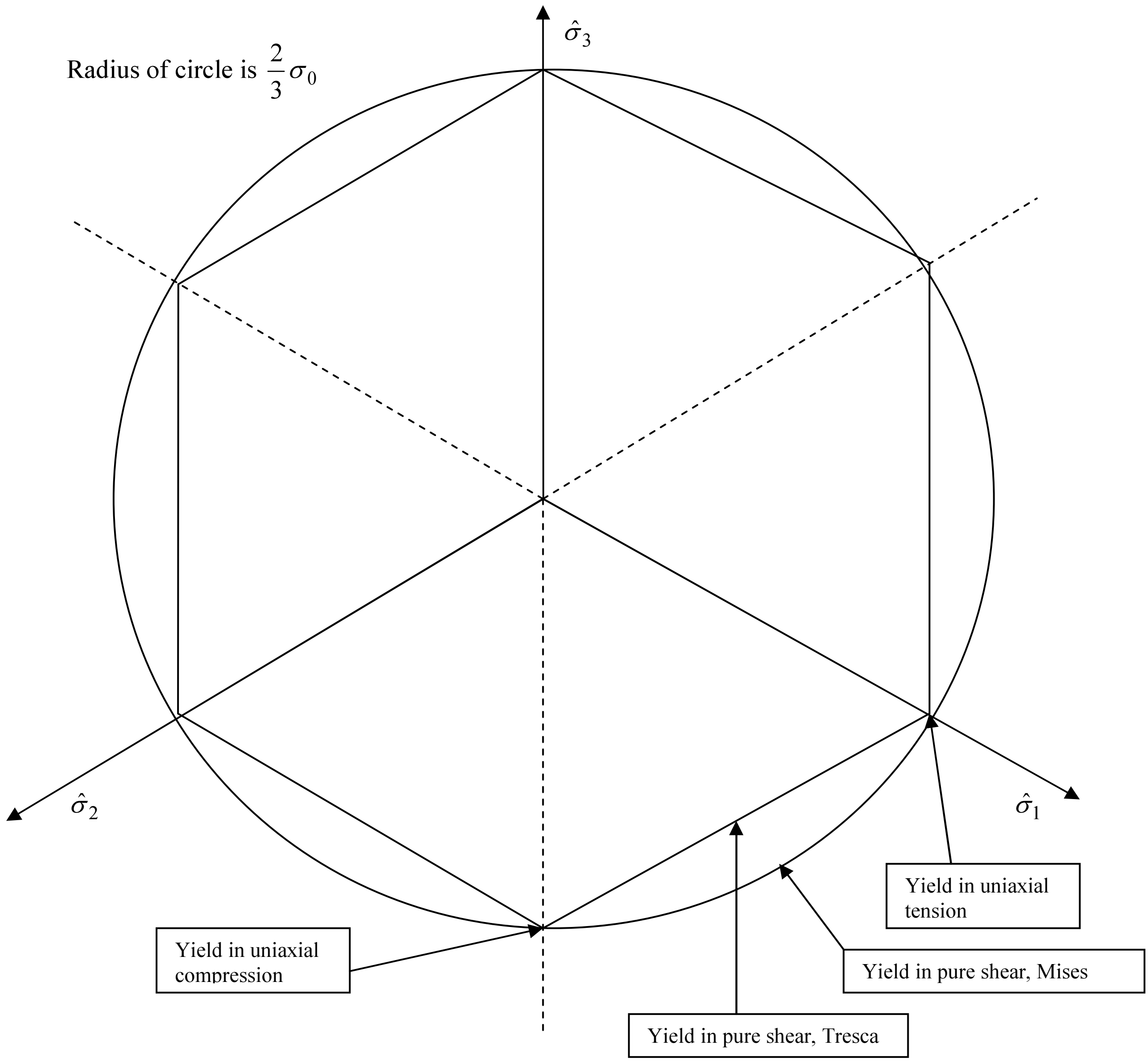
By symmetry it must be the point on the $\hat{\sigma}_1$ axis where it intersects the circle. Since, from the above discussion, the circle is of radius $(2/3)\sigma_0$, this implies that yield occurs for $\hat{\sigma}_1 = (2/3)\sigma_0$. Is this right? At this point we have (see diagram)

$\hat{\sigma}_2 = \hat{\sigma}_3 = -\hat{\sigma}_1/2$. Hence, the Mises stress is $\bar{\sigma} = \sqrt{\frac{3}{2}\left(\hat{\sigma}_1^2 + 2 \times \left(-\frac{\hat{\sigma}_1}{2}\right)^2\right)} = \frac{3}{2}\hat{\sigma}_1$, and

hence $\hat{\sigma}_1 = (2/3)\sigma_0$ at yield, as required.

An even simpler derivation is that $\sigma_1 = \sigma_0$ for yield in uniaxial tension, and $\sigma_H = (\sigma_1 + 0 + 0)/3 = \sigma_1/3$, so that $\hat{\sigma}_1 = \sigma_1 - \sigma_1/3 = (2/3)\sigma_0$.

**Figure 5: The Yield Curves in the Deviatoric Stress Plane**




Qu.: Why is it simpler to think in terms of the deviatoric stress plane?

Because,

- The yield surface is a circle, or a regular hexagon, and,
- ***The same yield surface applies for any state of stress***.

Qu.: Why is it simpler?

The degree of freedom which is missing from Figure 5 is the hydrostatic stress. But whatever the hydrostatic stress, the yield curve is just the same on the cross-section of Figure 1 defined by $\sigma_H$ = constant. Using the deviatoric plane makes maximum use of the natural symmetry of the 3D yield surface in Figure 1.

Qu.: What does the Tresca yield surface look like on the deviatoric plane?

The Tresca yield surface is a hexagon on the deviatoric plane, whose six vertices coincide with the Mises yield points for uniaxial tension or uniaxial compression (see diagram).

Qu.: Why?

Well, we know that the Tresca and Mises yield criteria coincide for uniaxial tension or compression. So the Tresca surface must pass through the six points where the Mises circle intersects the axes. But the Tresca stress is $\sigma_{\max} - \sigma_{\min} = \hat{\sigma}_{\max} - \hat{\sigma}_{\min}$ and hence varies linearly across the deviatoric stress plane (until a different principal stress becomes maximum or minimum). Hence, the Tresca surface must be a straight line between each of the six pairs of adjacent points where it coincides with Mises, i.e., a regular hexagon as illustrated.

Qu.: Does the Tresca stress in uniaxial stress check out?

Yes, of course. Since we have (say) $\hat{\sigma}_1 = (2/3)\sigma_0$ and $\hat{\sigma}_2 = \hat{\sigma}_3 = -\hat{\sigma}_1/2 = -\sigma_0/3$, we get $\sigma_{Tresca} = \hat{\sigma}_{\max} - \hat{\sigma}_{\min} = \frac{2}{3}\sigma_0 - \left(-\frac{1}{3}\sigma_0\right) = \sigma_0$, as it should.

Qu.: What points on the deviatoric plane represent yield in pure shear?

Pure shear has two equal and opposite principal stresses, and the third zero, and hence one example is $\hat{\sigma}_1 = -\hat{\sigma}_3 > 0$ and $\hat{\sigma}_2 = 0$. This pure shear yield point thus lies midway between the point of yield in uniaxial tension in the 1-direction and uniaxial compression in the 3-direction (see diagram).

Hence, there are six points of pure shear, each midway between the vertices. These correspond to xy, zx and yz shears, each of which can be of either sign.

Qu.: How is the difference between the Mises and Tresca shear yield criteria evident on the deviatoric plane?

The difference between the Mises and Tresca shear yield criteria is evident from the different distances at which the radial line representing pure shear, $\hat{\sigma}_1 = -\hat{\sigma}_3$, say, intersects the Mises circle and the Tresca hexagon. From the diagram, the difference in the two distances is clearly a factor of $\cos 30^o = \sqrt{3}/2$, i.e.,

$$\frac{TrescaYield}{MisesYield} = \frac{\sqrt{3}}{2}$$

- which agrees with the algebraic derivation in session 8b.

Qu.: What is the maximum possible difference between Tresca and Mises stresses?

In session 8b it was stated without proof that the maximum Mises:Tresca ratio was that for pure shear, i.e., $\sqrt{3}/2$. This is now obvious from the circle/hexagon diagram, which constitutes a proof.