Tutorial Session 8b: T72S01
Basic Plasticity: Part 2 – Yield Criteria, Stress Invariants & Constraint

Relates to T72S01 Knowledge & Skills 1.12

Last Update: 18/2/14

Shear yield strength (Tresca & Mises); Plastic incompressibility; Stress invariants; Constraint; Mises plane strain yield strength;

Grey shaded parts are beyond SQEP requirements

Qu.: What is the yield stress in shear given the tensile yield stress?

Tresca: $\tau_y = \sigma_y / 2$ Mises: $\tau_y = \sigma_y / \sqrt{3}$

Hence, Mises predicts yield in shear at 15% higher load than Tresca.

Proof:

Mises: $\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$. In pure shear, $\sigma_1 = -\sigma_2 = \tau$

and hence $\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\tau - (-\tau))^2 + (-\tau - 0)^2 + (0 - \tau)^2\right]^{1/2} = \frac{\tau}{\sqrt{2}}[4 + 1 + 1]^{1/2} = \sqrt{3}\tau$ and this equals σ_y at yield according to the Mises criterion, hence $\tau_y = \sigma_y / \sqrt{3}$. **QED**.

Tresca: Since, in pure shear, $\sigma_1 = -\sigma_2 = \tau$ we immediately have $\sigma_{Tresca} = \tau - (-\tau) =$ 2τ and this equals σ_y at yield according to the Tresca criterion, hence $\tau_y = \sigma_y / 2$.
QED.

Qu.: Why does $\sigma_1 = -\sigma_2 = \tau$ in pure shear?

Recall the transformation of stresses by rotation (earlier notes). In the case of a 45-degree rotation of a pure shear state this gives,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \tau & \tau \\ \tau & -\tau \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} \qquad \text{QED.}$$

Qu.: Is there any stress state for which Tresca and Mises differ by more than 15%?

No (see notes on the Deviatoric Stress Plane, next session).

Qu.: Is there any stress state other than pure shear for which the maximum difference between Tresca and Mises of 15% is realised?

Yes. We can add an arbitrary hydrostatic stress without affecting the Tresca or Mises equivalent stresses.

Plasticity in any state of plane strain achieves this maximum difference between Tresca and Mises. This is because plastic flow in plane strain occurs for a stress state which is pure in-plane shear plus a hydrostatic stress. The proof follows…

Qu.: What is the most general plastic plain strain state of stress?

First note that plane strain means that, for isotropic elasticity,

$$E\varepsilon_3 = \sigma_3 - \nu(\sigma_1 + \sigma_2) = 0 \Rightarrow \sigma_3 = \nu(\sigma_1 + \sigma_2)$$

where σ_1 etc. are the principal stresses (not necessarily in the order of magnitude), and the 3 direction is out-of-plane.

Hence, for plastic behaviour, for which $\nu = 1/2$, we have $\sigma_3 = (\sigma_1 + \sigma_2)/2$. For the rightly suspicious, this can be deduced more properly from the plasticity flow rule – see next session – without going via the elastic result. Strictly the "stresses" are really stress rates, or stress increments. Alternatively they are indeed the stresses if there is no elastic behaviour prior to plasticity.

The general 2D plane strain stress state for such idealised plasticity is thus,

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \frac{1}{2}(\sigma_1 + \sigma_2) \end{pmatrix} = \frac{1}{2}(\sigma_1 - \sigma_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}(\sigma_1 + \sigma_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This should be recognised as a pure shear plus a pure hydrostatic stress, with $\tau = (\sigma_1 - \sigma_2)/2$ and $\sigma_H = (\sigma_1 + \sigma_2)/2$. Rotating the coordinate system by 45° in-plane the stress state can be written,

$$\tau\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sigma_H\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The most general state of stress in plane strain conditions when plastic flow is occurring (or whenever $\nu = 0.5$) is a hydrostatic stress plus an in-plane shear

But if yielding is occurring, as assumed, then the shear stress must be at the yield stress – i.e., either $\tau_y = \sigma_y / 2$ for Tresca, or $\tau_y = \sigma_y / \sqrt{3}$ for Mises. Hence,

The most general state of stress in plane strain conditions when plastic flow is occurring is a hydrostatic stress plus an in-plane shear of magnitude τ_y

Qu.: What does this imply for plane strain yield loads?

A corollary of the above is,

If plane strain conditions prevail then the yield load based on the Mises criterion will always be a factor $2/\sqrt{3}$ larger than the yield load based on Tresca (i.e. 15% bigger).

You will this factor of $2/\sqrt{3}$ between Mises and Tresca solutions frequently in compendia of 'collapse', or yield, loads (or reference stresses).

Qu.: What is the effective yield stress for an x-load in plane strain?

Consider uniaxial loading in plane strain. For the Tresca criterion the applied stress required to cause yielding is simply the same yield stress as for plane stress, σ_0. However, for the Mises criterion it follows from the above that the stress required will be $\frac{2}{\sqrt{3}}\sigma_0 = 1.155\sigma_0$. Loosely speaking, the effective Mises yield stress in plane strain is, $\frac{2}{\sqrt{3}}\sigma_0 = 1.155\sigma_0$ whereas the Tresca yield stress is unchanged. Of course, the Mises stress at yield is σ_0 according to the Mises yield criterion.

Qu.: Is plane stress also equivalent to a shear plus a hydrostatic stress?

No.

It is obvious from, say, the case of uniaxial stressing, which is a special case of plane stress, that the Tresca and Mises can be equal – but will not be in general plane stress. Generally in plane stress,

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 - \sigma_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}(\sigma_1 + \sigma_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & -\sigma_H \end{pmatrix} + \sigma_H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{rotating gives } \rightarrow \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & -\sigma_H \end{pmatrix} + \sigma_H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

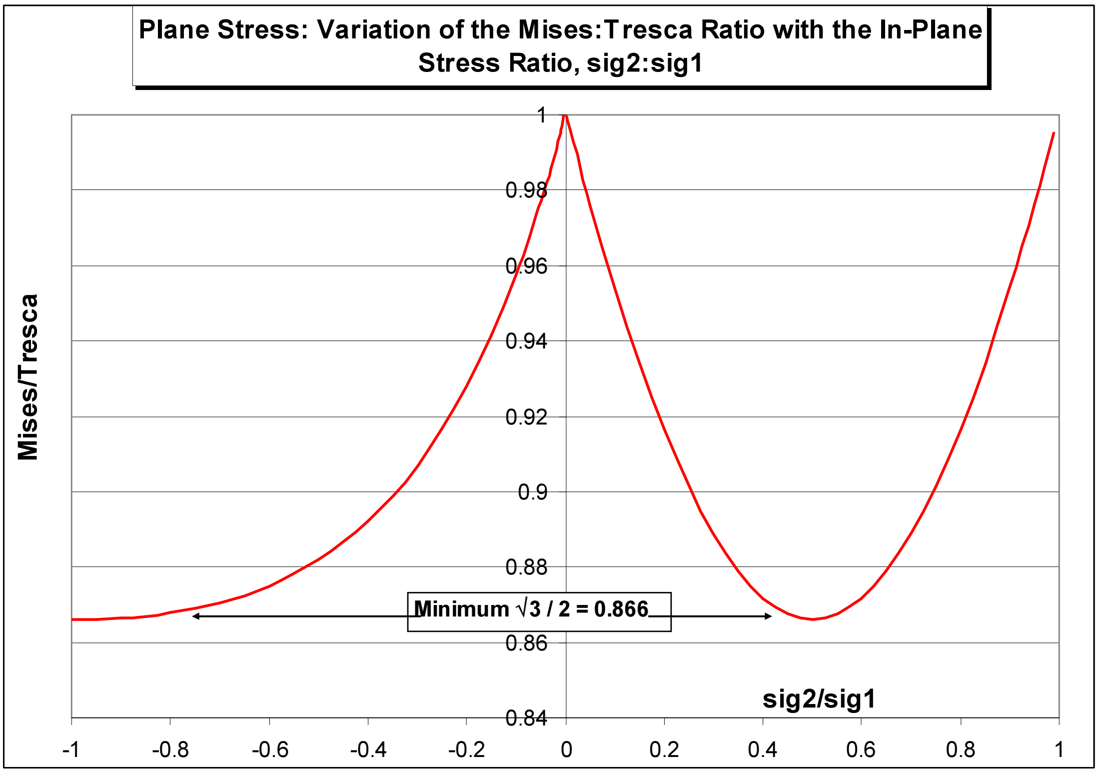
where $\tau = (\sigma_1 - \sigma_2)/2$ and $\sigma_H = (\sigma_1 + \sigma_2)/2$ as before. But the first term is not pure shear – it is only pure shear in the *x,y* plane.

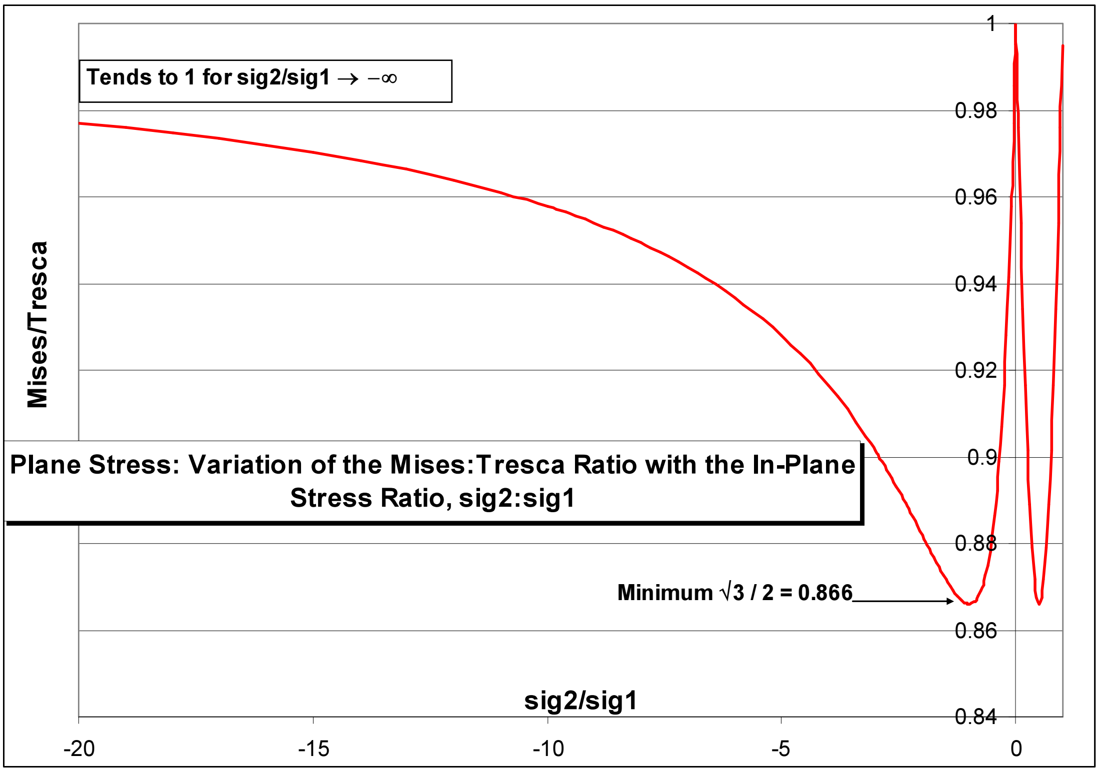
Assuming, without loss of generality, that $\sigma_1 > \sigma_2$ it follows that $\sigma_1 > \sigma_H > \sigma_2$ and hence that the Tresca stress is either σ_1 (if $\sigma_2 > 0$) or 2τ (if $\sigma_2 < 0$). The Mises stress is $\bar{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$. Hence, setting $\xi = \frac{\sigma_2}{\sigma_1}$ its allowed values are $-\infty < \xi < 1$ and,

For $\xi < 0$: $$\frac{\bar{\sigma}}{\sigma_T} = \frac{\sqrt{1 - \xi + \xi^2}}{1 - \xi}$$

For $\xi > 0$: $$\frac{\bar{\sigma}}{\sigma_T} = \sqrt{1 - \xi + \xi^2}$$

So the Mises:Tresca ratio can be anywhere in the range $\sqrt{3}/2$ to 1, see graphs below,


Plane Stress: Variation of the Mises:Tresca Ratio with the In-Plane Stress Ratio, sig2:sig1
Mises/Tresca
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
Minimum √3 / 2 = 0.866
sig2/sig1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Tends to 1 for sig2/sig1 → −∞
Mises/Tresca
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
Plane Stress: Variation of the Mises:Tresca Ratio with the In-Plane Stress Ratio, sig2:sig1
Minimum √3 / 2 = 0.866
sig2/sig1
-20
-15
-10
-5
0

Qu. Does the √3 factor between tensile and shear yield also apply to UTS versus ultimate shear strength?

No. Recall that the UTS is really a necking instability, caused by the reduction of the cross-section on straining. But shear does not cause any cross-section reduction. Consequently the ultimate shear strength (USS) is greater than UTS/√3, though it is still generally less than the UTS. Typical values for structural steels are UTS/1.2 or UTS/1.3 (see, for example, R66 or R51).

Qu.: What is a scalar?

A scalar is a quantity whose value does not change if the coordinate system is changed, e.g., length or mass (providing we're not relativistic). A scalar is a rank 0 tensor.

Qu.: How many independent scalar quantities can be made out of a stress tensor?

Three – this is obvious since the three principal stresses are an example of such a set of independent scalars. But there are more useful alternatives…

Qu.: How do we make scalars out of a tensor?

By contracting the indices. So, the most obvious scalar is just the trace of the tensor, i.e.,

$$J_1 \equiv \sigma_{ii} = Tr(\sigma) = 3\sigma_H = \sigma_1 + \sigma_2 + \sigma_3$$

where σ_H, the average of the diagonal components (the direct stresses) is the hydrostatic stress. (NB: Recall that repeated indices are summed).

The next most obvious is,

$$Z_2 \equiv \sigma_{ij}\sigma_{ji} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

The determinant of the tensor also provides a scalar,

$$J_3 \equiv \|\sigma\| = \sigma_1\sigma_2\sigma_3$$

Qu.: Why is the determinant a scalar?

Because $\|AB\| = \|A\|\|B\|$ and the tensor transforms as $\sigma' = Q^T \sigma Q$ under the orthogonal matrix Q representing the rotation. Hence, $\|\sigma'\| = \|Q^T\|\|\sigma\|\|Q\|$. But $Q^T = Q^{-1}$ and hence $\|Q\| = 1$ and hence $\|\sigma'\| = \|\sigma\|$. QED. Another proof is to note that the determinant can be written $\|\sigma\| = \varepsilon_{ijk}\sigma_{1i}\sigma_{2j}\sigma_{3k} = \frac{1}{3!}\varepsilon_{lmn}\varepsilon_{ijk}\sigma_{li}\sigma_{mj}\sigma_{nk}$, where ε_{ijk} is the alternating tensor and hence, being a tensor contraction, is a scalar.

Qu.: Can any function of the principal stresses be expressed in terms of J_1, Z_2, J_3?

Yes, because the principal stresses can be found from J_1, Z_2, J_3. It is usual to define,

$$J_2 \equiv -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

It is not obvious that this is a scalar, but it is. Note that it can be written,

$$J_2 = \frac{1}{2}\left(Z_2 - J_1^2\right) = \frac{1}{2}\left(\sigma_{ij}\sigma_{ji} - (\sigma_{ii})^2\right)$$

From which it follows that it is indeed a scalar. Given J_1, Z_2 and J_3, or equivalently, given J_1, J_2 and J_3, the principal stresses are the roots of the equation,

$$\lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0$$

Qu.: Could we use the triple contraction in place of the determinant?

Yes. Put $Z_3 = \sigma_{ij}\sigma_{jk}\sigma_{ki} = \sigma_1^3 + \sigma_2^3 + \sigma_3^3$. A bit of algebra shows that we can recover J_3 as,

$$J_3 = \frac{1}{3}\left(Z_3 - J_1^3 - 3J_1J_2\right)$$

Qu.: What is the most general yield criterion for an isotropic material which is plastically incompressible?

If an isotropic material deforms plastically at constant volume, i.e., if it is plastically incompressible and hence has $\varepsilon_1^P + \varepsilon_2^P + \varepsilon_3^P = 0$, then it follows that the yield criterion cannot depend upon the hydrostatic stress. This is because, since the material is isotropic, a purely hydrostatic stress, being isotropic, could only give rise to an isotropic (i.e. hydrostatic) plastic strain. But the latter is zero, so hydrostatic stress cannot cause plasticity.

It is convenient to redefine all the stress invariants in terms of the deviatoric stress tensor defined by subtracting off the hydrostatic stress,

$$\hat{\sigma}_{ij} \equiv \sigma_{ij} - \sigma_H\delta_{ij}$$

Replacing σ_{ij} with $\hat{\sigma}_{ij}$ in all the definitions gives,

$$J_1' = \hat{\sigma}_{ii} = 0$$

$$J_2' = \frac{1}{2}\left(Z_2' - J_1'^2\right) = \frac{Z_2'}{2} = \frac{1}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij} = \frac{1}{2}\left(\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2\right)$$

$$J_3' = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \frac{Z_3'}{3} = \frac{1}{3}\left(\hat{\sigma}_1^3 + \hat{\sigma}_2^3 + \hat{\sigma}_3^3\right)$$

The last follows from $J_3' = \frac{1}{3}\left(Z_3' - J_1'^3 - 3J_1'J_2'\right)$, as observed above, together with $J_1' = 0$.

Note that J_2' is equivalent to the Mises stress, specifically $J_2' = \frac{\bar{\sigma}^2}{3}$. The most general yield criterion is thus,

$$f\left(J_2', J_3'\right) = \sigma_y$$

The Mises criterion is the special case that *f* is independent of J_3'. In this notation the Mises criterion is,

$$J_2' = \tau_y^2 = \frac{\sigma_y^2}{3}$$

Consequently, Mises flow theory is often called "J_2' flow theory" in texts.

Qu.: What is the physical interpretation of the Mises yield criterion?

For isotropic media the Mises criterion is equivalent to a criterion in which yielding initiates when the elastic strain energy associated with distortion, i.e., with deviatoric strains, reaches a critical value. Specifically, the elastic distortion energy density, $\hat{\xi}$, is related to the Mises stress by $\hat{\xi} = \frac{1+\nu}{3E}\bar{\sigma}^2$.

Qu.: What is "constraint"?

"Constraint" is a slippery concept, but is extremely important. Metallurgists use the term frequently, but in an intuitive manner and never define it. There is no unique numerical definition. Nevertheless, it is a crucially important concept. It is constraint which gives rise to highly triaxial states of stress (large hydrostatic stress), and it is often this which exacerbates fracture effects or crack initiation.

The essence of constraint is that, under a given load, lateral contraction is constrained. This leads to reduced strains but elevated lateral stresses, and hence increased hydrostatic stress. A constrained geometry/loading has increased hydrostatic stress compared with an unconstrained geometry/loading, other things being equal. Constraint can reduce the likelihood of yielding at a given load, but increase the likelihood of fracture. High constraint tends to lower the effective fracture toughness and increase the rate of crack growth due to (say) creep. It also greatly increases the likelihood of creep crack initiation (Spindler fraction / reheat cracking).

Qu.: Is constraint equally important in elasticity and plasticity (or creep)?

No.

Constraint has an effect under both elastic and plastic conditions, but is much more important in plasticity (and creep). A simple example illustrates the point: consider what happens if, under an applied x-stress, the y- and z-components of strain are constrained to be zero.

Elastic Case

Assuming isotropy, we have $E\varepsilon_y = \sigma_y - \nu(\sigma_x + \sigma_z) = 0$, and similarly for the z-strain, which gives $\sigma_y = \sigma_z = \frac{\nu}{1-\nu}\sigma_x$ and hence $\sigma_H = \frac{1}{3}\left(1 + \frac{2\nu}{1-\nu}\right)\sigma_x$, or,

$$\sigma_H = 0.62\sigma_x$$

in the elastic case with $\nu = 0.3$. Note that $\sigma_H = 0.33\sigma_x$ under uniaxial tension, so the constraint has roughly doubled the hydrostatic stress in the elastic case. The elastic x-strain is found from $E\varepsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z) = \left(1 - \frac{2\nu^2}{1-\nu}\right)\sigma_x$ and is thus,

$$\varepsilon_x = 0.74 \text{ times the x-strain without lateral constraint,}$$

if $\nu = 0.3$. Hence, the strain is reduced but the hydrostatic stress is increased by the constraint.

Plastic Case

Now consider the plastic case. Since $\varepsilon_x^P + \varepsilon_y^P + \varepsilon_z^P = 0$ it immediately follows that, if the plastic y- and z-strains are zero, the plastic x-strain must also be zero. Hence, constraint has a more marked effect in plasticity since the x-strain is reduced to zero, rather than merely to 74% as in the elastic case. This can also be derived from the elastic formulae $E\varepsilon_x = \left(1 - \frac{2\nu^2}{1-\nu}\right)\sigma_x$ by using the incompressible value $\nu = 0.5$. We can figure out what the hydrostatic stress must be in the 'plastic' case because we have actually stopped plasticity occurring at all, however high the load. A heuristic indication is provided by the elastic formula $\sigma_y = \sigma_z = \frac{\nu}{1-\nu}\sigma_x$ in the incompressible limit, $\nu = 0.5$, which gives $\sigma_H = \sigma_x = \sigma_y = \sigma_z$. More correctly, the fact that plastic straining is prevented at any load must mean that the equivalent stress is zero, and this must mean that $\sigma_x = \sigma_y = \sigma_z$. So the hydrostatic stress is tripled rather than doubled as in the elastic case, and we see that constraint has an even greater effect in plasticity (or creep) than in elasticity.

Qu.: Does constraint affect the yield load?

Yes. This is merely because it changes the lateral stresses, which contribute to the yield condition. In the extreme case of full constraint in the y- and z-directions, yield will never occur for any x-load. We have already seen this in the case of uniaxial stress in plane strain, for which the applied stress must be $2/\sqrt{3}$ times the yield stress to cause yielding.