SQEP Tutorial Session 7: T72S01
Relates to Knowledge & Skills: 1.10
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Formulation of the general continuum elasticity problem; Formulation in terms of displacements or use of the compatibility equations; Reduction to a single scalar equation in 2D (Airy functions); Plane stress and plane strain problems; Example solutions.

Qu.: How do we formulate the general elasticity problem?

Well, what do we know?

We know how to formulate the requirements of equilibrium

We know how stress and strain are related

We know how strain is defined in terms of displacements

Qu.: What are the equilibrium equations?

$$\sigma_{ij,j} = -b_i \qquad \text{(}b_i\text{ is the applied force per unit volume)} \qquad (1)$$

Qu.: Do these equilibrium equations allow us to solve a problem?

No, because there are six stress components and only three equations.

Qu.: What are the (elastic) relations between stress and strain components?

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad \text{(generally anisotropic)} \qquad (2)$$

Qu.: Do equations (1) and (2) allow us to solve a problem?

No, because whilst (2) provides six more equations it also introduces six more unknowns, namely the six components of strain. So equations (2) get us no further forward on their own.

Qu.: How are the strain components expressed in terms of displacements?

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (3)$$

Qu.: Do equations (1), (2) and (3) allow us to solve a problem?

Yes, finally. There are six equations (3), but they introduce only three more unknowns (the three displacements). This is a net benefit of three, which is just what is needed.

Substituting (3) into (2) gives the six stress components in terms of the derivatives of the three displacements. Substitution into (1) then gives three equations in the second derivatives of the three displacements.

Qu.: So how does this work out?

It gives, (4)

$$\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)u_{x,xx} + \nu u_{y,yx} + \nu u_{z,zx}\right] + \frac{E}{2(1+\nu)}\left[u_{x,yy} + u_{y,xy} + u_{x,zz} + u_{z,xz}\right] = b_x$$

and two similar equations for the y and z directions.

Qu.: In this context, how does finite element analysis work?

Finite element analysis is essentially solving the three equations like (4), albeit formulated in a different manner. The discrete finite elements, their shape functions, etc., effectively mean that the LHS of (4) gets re-written in the form $(K)\overline{u}$.

Qu.: So, we don't need the dreaded "compatibility equations"?

Nope.

Qu.: So why does anyone bother with the compatibility equations?

Because they provide a way of formulating the elasticity problem without involving displacements, that is, in terms of stress and strain alone.

Qu.: How is this possible?

The compatibility equations are a set of second order differential equations in the strain components which are equivalent to the existence of displacements such that equations (3) hold. In maths-speak, they are the integrability conditions for equations (3). In 3D Cartesian coordinates the compatibility equations are, (5)

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy}, \qquad \varepsilon_{xx,zz} + \varepsilon_{zz,xx} = 2\varepsilon_{xz,xz}, \qquad \varepsilon_{zz,yy} + \varepsilon_{yy,zz} = 2\varepsilon_{zy,zy}$$

$$\varepsilon_{xy,xz} + \varepsilon_{xz,xy} = \varepsilon_{yz,xx} + \varepsilon_{xx,yz}, \quad \varepsilon_{yx,yz} + \varepsilon_{yz,yx} = \varepsilon_{xz,yy} + \varepsilon_{yy,xz}, \quad \varepsilon_{zy,zx} + \varepsilon_{zx,zy} = \varepsilon_{yx,zz} + \varepsilon_{zz,yx}$$

These are easily checked by substituting equations (3) into (5). This means that equations (3) imply equations (5). What is less obvious, but true, is that equations (5) also imply equations (3), i.e., that displacements exist such that equations (3) are true.

Qu.: So, how is the general 3D elasticity problem formulated?

You have a choice. You can formulate in terms of displacements and use,

Equations (1), (2) and (3)

Or you can formulate without displacements being involved and use,

Equations (1), (2) and (5)

Qu.: How does this simplify for 2D problems?

One of the simplifications is that instead of six compatibility equations there is only one. It is,

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} = 2\varepsilon_{xy,xy} = \gamma_{,xy} \tag{6}$$

The equations of equilibrium and Hooke's law in 2D plane stress become,

$$\sigma_{x,x} + \tau_{,y} = 0, \quad \sigma_{y,y} + \tau_{,x} = 0 \tag{7}$$

$$E\varepsilon_x = \sigma_x - \nu\sigma_y, \quad E\varepsilon_y = \sigma_y - \nu\sigma_x, \quad G\gamma = \tau \tag{8}$$

Qu.: Are there any other simplifications of the 2D equations?

Yes. A further simplification is possible by introducing an Airy function, φ. This acts like a 'potential' function for the stress field. The stresses are given in terms of the Airy function by,

$$\sigma_x = \varphi_{,yy} \qquad \sigma_y = \varphi_{,xx} \qquad \sigma_{xy} = \tau = -\varphi_{,xy} \tag{9}$$

In terms of the Airy function, the equations of equilibrium, Equs.(7), become identities and we need not worry about them any more. [In fact, the existence of an Airy function is ensured by the equilibrium equations, i.e., equs.(9) are the integrability conditions of equs.(7)].

But the real prize is that by substituting (9) into (8), and substituting the resulting expressions for the strains into (6), we get simply,

$$\nabla^4 \varphi = 0 \tag{10}$$

where $\nabla^4 = \left(\nabla^2\right)^2 = \left(\partial_x^2 + \partial_y^2\right)^2$. Equ.(10) is called the biharmonic equation, though in this context it is also called Airy's equation.

This shows that the 2D problem reduces to just one equation in just one unknown function, the Airy function, φ. This is a huge simplification. However, it is a differential equation of fourth order.

If we can solve for the Airy function, the stresses will be given by Equs.(9), and the strains by Equs.(8).

Qu.: How can the 2D problem be formulated in other coordinate systems?

Having derived Equ.(10), to express a problem in 2D polars is just a matter of knowing what ∇^2 is in 2D polars. Without proof it is,

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \tag{11}$$

If the problem is also axisymmetric, then this becomes,

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r \tag{12}$$

In which case, Airy's equation, (10), becomes,

$$\left[\partial_r^4 + \frac{2}{r}\partial_r^3 - \frac{1}{r^2}\partial_r^2 + \frac{1}{r^3}\partial_r\right]\varphi = 0 \tag{13}$$

It is straightforward to show by substitution that the general solution to (13) is,

$$\varphi = A + Br^2 + C\log r + Dr^2 \log r \tag{14}$$

Qu.: What are the stresses in terms of the Airy Function in 2D polars?

In 2D polars, equations (9) for the stresses in terms of the derivatives of the Airy function become,

$$\sigma_r = \frac{1}{r^2}\varphi_{,\theta\theta} + \frac{1}{r}\varphi_{,r} \quad \text{and} \quad \sigma_\theta = \varphi_{,rr} \quad \text{and} \quad \tau_{r\theta} = \frac{1}{r^2}\varphi_{,\theta} - \frac{1}{r}\varphi_{,r\theta} \tag{15a}$$

If the problem is also axisymmetric, then these become,

$$\sigma_r = \frac{1}{r}\varphi_{,r} \quad \text{and} \quad \sigma_\theta = \varphi_{,rr} \quad \text{and} \quad \tau_{r\theta} = 0 \tag{15b}$$

Qu.: Do these equations apply for plane stress or plane strain?

Yes.

Either.

As written, Equ.(8) is for plane stress. But the plane strain problem requires no extra effort. In plane strain there is no change to the equations (1, 3, 6, 7, 9), but the stress-strain relations become,

$$E\varepsilon_x = \sigma_x - \nu\sigma_y - \nu\sigma_z, \qquad E\varepsilon_y = \sigma_y - \nu\sigma_x - \nu\sigma_z, \qquad E\varepsilon_z = \sigma_z - \nu\sigma_x - \nu\sigma_y = 0 \tag{16}$$

The latter gives $\sigma_z = \nu(\sigma_x + \sigma_y)$, so that the first two become,

$$E\varepsilon_x = (1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y \quad \text{and} \quad E\varepsilon_y = (1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x \tag{17}$$

But defining, $E' = \dfrac{E}{1-\nu^2}$ and $\nu' = \dfrac{\nu}{1-\nu}$ these become, (18)

$$E'\varepsilon_x = \sigma_x - \nu'\sigma_y \quad \text{and} \quad E'\varepsilon_y = \sigma_y - \nu'\sigma_x \tag{19}$$

These are formally identical to the plane stress equations, (8). Hence,

The solution to any 2D plane stress problem also provides the solution to the corresponding 2D plane strain problem simply by making the replacements $E \to E'$ and $\nu \to \nu'$