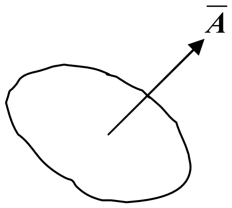
SQEP Tutorial Session 5: T72S01
Relates to Knowledge & Skills 1.5, 1.8

Last Update: 11/12/13

Force on an element of area; Definition of principal stresses and strains; Definition of Tresca and Mises equivalent stresses; Mohr's circles; Hydrostatic and deviatoric stresses;

Qu.: How is an element of area described?

An element of area is described by a vector. The magnitude of the vector equals the area. The direction of the vector is the normal to area element. Note that this means that area elements have an orientation, i.e., an outside and an inside.



Qu.: Given the stress matrix, what is the force acting on an element of area?

If the area happened to be oriented in the x-direction, the three components of the force would be, by definition, $A\sigma_{xx}$, $A\sigma_{xy}$ and $A\sigma_{xz}$. For any other orientation of the area, if A_x is the component of the area-vector in the x-direction, then this component of the area produces force components of $A_x\sigma_{xx}$, $A_x\sigma_{xy}$ and $A_x\sigma_{xz}$. Similarly, the component of the area-vector in the y-direction, A_y, produces force components $A_y\sigma_{yx}$, $A_y\sigma_{yy}$ and $A_y\sigma_{yz}$. The total force in the x-direction is thus,

$$F_x = A_x\sigma_{xx} + A_y\sigma_{yx} + A_z\sigma_{zx} \equiv A_i\sigma_{ix}$$

And similarly in the other directions, so the overall result is expressed compactly as,

$$F_j = A_i\sigma_{ij}$$

(Note the summation convention for repeated indices). If we adopt vector/matrix notation this can also be written,

$$\overline{F} = (\sigma)\overline{A} \qquad \text{(because } \sigma \text{ is symmetric)} \qquad (1)$$

Qu. What is the direct stress in an arbitrary direction, $\hat{n}$?

For a unit area, we can put $\overline{A} = \hat{n}$, where $\hat{n}$ is the unit vector normal to the area element. The force acting on this unit area is thus $\overline{F} = (\sigma)\hat{n}$, so the component of this force in the direction of $\hat{n}$ is $F_n = \hat{n}^T(\sigma)\hat{n}$. But this is the normal force on a unit area, i.e., it is the direct stress in the direction $\hat{n}$. That is $\sigma_{nn} = \hat{n}^T(\sigma)\hat{n}$, or, in component notation,

$$\sigma_{nn} = \hat{n}_i\sigma_{ij}\hat{n}_j \qquad (2)$$

Qu.: How do all the stress components change due to a rotation of the coordinate system?

Restricting attention to a rotation in the (x,y) plane, the stress components in the rotated coordinate system are given by,

$$\begin{pmatrix} \sigma'_x & \tau' \\ \tau' & \sigma'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

(this will be derived in Session 6) and hence,

$$\begin{aligned} \sigma'_x &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau\cos\theta\sin\theta \\ \sigma'_y &= \sigma_y \cos^2\theta + \sigma_x \sin^2\theta - 2\tau\cos\theta\sin\theta \\ \tau' &= \tau\cos 2\theta + (\sigma_y - \sigma_x)\cos\theta\sin\theta \end{aligned} \tag{4}$$

Qu.: What is meant by the "principal stresses"?

Supposing the element of area is rotated until the force acting on it is normal to the surface, i.e., $\overline{F}$ is parallel to $\overline{A}$ (i.e., normal to the element of area). Then the stress in this direction is one of the principal stresses.

Qu.: How many principal stresses are there?

Three.

The directions in which the principal stresses act are mutually orthogonal, i.e. they form a Cartesian coordinate system. So, it is always possible to rotate the initial (x,y,z) coordinate system to align with the principal axes. The proof of these assertions will be given in Session 6.

Qu.: What does the stress matrix look like in principal coordinates?

Suppose (x', y', z') are the principal axes. The forces acting on the faces of a unit cube oriented parallel to these axes are all perpendicular to the faces. In other words, there are only direct stresses, all the shear stresses being zero. So the stress matrix is diagonal,

$$(\sigma) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{5}$$

NB: It is conventional to order the principal stresses so that $\sigma_1 > \sigma_2 > \sigma_3$, but their order in the matrix may not be as indicated above.

Qu.: Can the principal axes be defined by the vanishing of the shear stresses?

Yes. The principal axes are the only axes for which the shears all disappear.

To be pedantic, if two of the principal stresses are equal then there is a degree arbitrariness in the orientation of the principal axes (degeneracy).

Qu.: Are there "principal strains" also?

Yes. For an isotropic, elastic material this follows from the 3D Hooke's law, since the vanishing of the shear stresses implies the vanishing of the shear strains, i.e., because $G\gamma_{x'y'} = \sigma_{x'y'}$, etc. Moreover, in this case the principal stress axes and the principal strain axes are the same.

Qu.: Do principal axes always exist even for anisotropic materials?

Yes.

The existence of principal stress directions (in terms of which all shear stresses are zero) follows as a mathematical consequence of the symmetry of the stress tensor.

Exactly the same is true for strain. The existence of principal strain directions (in terms of which all shear strains are zero) follows as a mathematical consequence of the symmetry of the strain tensor.

Consequently, principal axes for both stress and strain always exist, irrespective of isotropy or elasticity.

Qu.: Are the principal axes always the same for stress and strain?

No.

In linear elasticity for an isotropic material the principal axes for stress coincide with those for strain, since the vanishing of shear stresses implies the vanishing of shear strains in this case.

However, for anisotropic materials the principal stress axes and the principal strain axes will generally be different.

Qu.: What are equivalent stresses?

Equivalent stresses are a convenient way of reducing the six components of the stress matrix to a single scalar measure of (roughly speaking) the severity of the stressing. More accurately, they express the severity of the stressing only in certain respects, such as the proximity to, or degree of, yielding, or the degree of distortion. Be aware, though, that other stress measures, independent of the equivalent stress, such as the hydrostatic stress, can strongly influence other issues (such as fracture).

Qu.: What is the Tresca stress?

The Tresca stress is simply twice the maximum shear stress for any orientation of the coordinate system. It is given in terms of the maximum and minimum principal stresses by,

$$\sigma_{Tresca} = \sigma_1 - \sigma_3 = \textit{twice maximum shear} \tag{6}$$

Qu.: Why is the maximum shear equal to $(\sigma_1 - \sigma_3)/2$?

Recall from Session 4 that a pure shear of τ corresponds to in-plane principal stresses of τ and $-\tau$. If we have in-plane principal stresses σ_1 and σ_2, then the 2D (in-plane) stress matrix can be expressed as the sum,

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_1 + \sigma_2 & 0 \\ 0 & \sigma_1 + \sigma_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sigma_1 - \sigma_2 & 0 \\ 0 & -(\sigma_1 - \sigma_2) \end{pmatrix} \tag{7}$$

But the first term on the RHS is an isotropic stress, and hence does not give rise to any shear when rotated (it is a purely hydrostatic stress). The second term, however, is of pure shear form – or rather it becomes so when rotated by 45^o – and the magnitude of the shear is just $(\sigma_1 - \sigma_2)/2$.

Of the three coordinate planes, the largest shear is therefore found between the maximum and the minimum principal stresses, i.e., $(\sigma_1 - \sigma_3)/2$. QED.

Qu.: What is the von Mises stress?

In terms of the principal stresses the Mises stress is given by,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{8}$$

In an arbitrary coordinate system, the Mises stress is,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right)\right]^{1/2} \tag{9}$$

The main motivation is that setting the Mises stress to the yield stress provides the best yield criterion for multi-grained, isotropic metals (well, structural steels, at least).

Qu.: Do the Mises or Tresca stresses depend upon the coordinate system used?

No. In other words, the Mises stress and the Tresca stress are scalars. This is proved in Session 8b.

Qu.: What is the hydrostatic stress?

The hydrostatic stress is the average of the three direct stresses, i.e., one-third of the trace of the stress matrix. Note that it is also a scalar, i.e., it is the same in all coordinate systems.

$$\sigma_H = Tr(\sigma)/3 = \sigma_{ii}/3 = (\sigma_1 + \sigma_2 + \sigma_3)/3 = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 \tag{10}$$

Qu.: If a stress state is changed by adding a purely hydrostatic stress, do the equivalent stresses change?

No.

Neither the Tresca stress nor the Mises stress change. This is because all three principal stresses increase by the same amount, i.e., by the hydrostatic stress added.

Qu.: What are the deviatoric stresses?

The deviatoric stresses are the on-diagonal (direct) stresses minus the hydrostatic stress. They are denoted by a 'hat' (caret): thus, $\hat{\sigma}_x = \sigma_x - \sigma_H$. The shear stresses are not affected. Hence,

$$\hat{\sigma}_{ij} = \sigma_{ij} - \sigma_H \delta_{ij} \qquad \text{or} \qquad \hat{\sigma} = \sigma - \sigma_H \mathrm{I} \tag{11}$$

where I is the unit matrix and the Kronecker symbol δ_{ij} are its components, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Qu.: What is the Mises stress in terms of the deviatoric stresses?

The answer is,

$$\bar{\sigma} = \left[\frac{3}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij}\right]^{1/2} \tag{12}$$

noting that the summation convention applies to both indices. So the Mises stress does not depend upon the hydrostatic stress, only upon the deviatoric stresses.

Qu.: What is the physical interpretation of the Mises yield criterion?

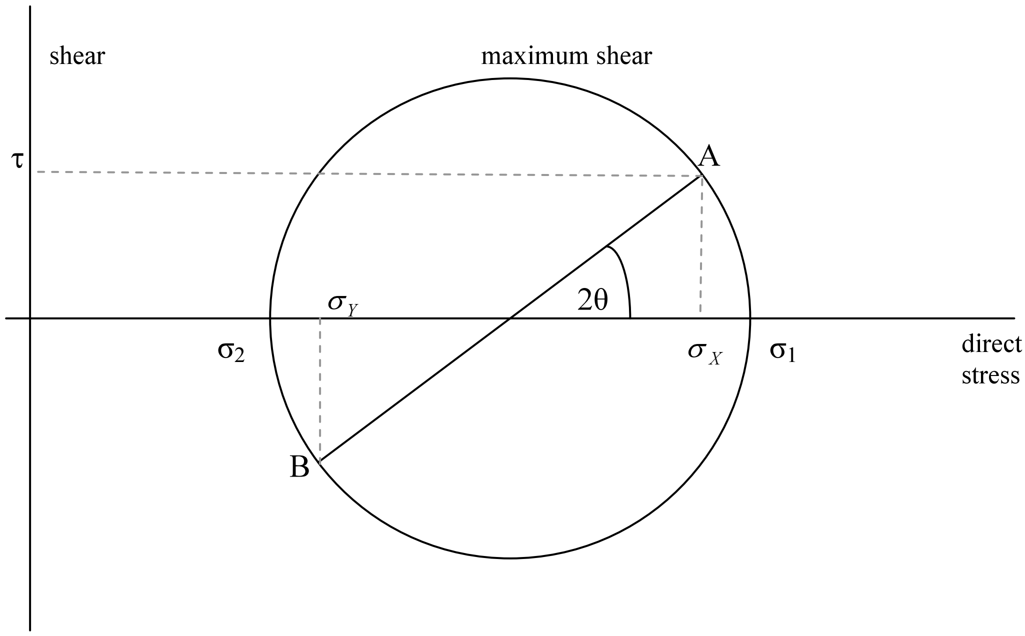
An interpretation for isotropic media is that the Mises criterion is equivalent to a criterion for which yielding initiates when the elastic strain energy associated with distortion, i.e. with deviatoric strains, reaches a critical value. Specifically, the elastic distortion energy density, $\hat{\xi}$, is related to the Mises stress by $\hat{\xi} = \frac{1+\nu}{3E}\bar{\sigma}^2$.

Qu.: What are Mohr's Circles?

Mohr's circles are a means of visualising the state of stress at a point. A vector field at a point can be visualised as an arrow. The Mohr's circles are the equivalent for the stress field. It is more complicated because stress is a second rank tensor rather than a vector. The Mohr's circles are drawn on a plot of shear stress (y) against direct stresses (x). The centre of a Mohr's circle always lies on the x-axis.

Qu.: How do you draw the Mohr's circle for plane stress?

Place the two principal stresses on the x-axis and draw the circle through them, and with its centre at their mid-point.



Qu.: So what's it good for?

The magnitudes of the two direct stresses and the shear stress in a coordinate system which is rotated by an angle θ from the principal axes are given by the points A and B. Note that the diameter AB is at an angle of 2θ to the x-axis. The x-coordinates of A and B are the two direct stresses, σ_x and σ_y, and the y-coordinate of A is the shear stress. (Note that the y-coordinate of B is an equal and opposite shear stress). From this it is readily seen that the stresses at an angle θ to the principal axes are,

$$\sigma_x = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$$

$$\sigma_y = \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \qquad (13)$$

$$\tau = \sigma_{xy} = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$$

These are consistent with the equations given above for a general rotation of coordinates from (x, y) to (x', y'), but for the special case that the initial coordinate system is the principal system.

Qu.: Where is the point of maximum *in-plane* shear?

Maximum shear must place A at the point with the largest y-coordinate, i.e., at the top of the circle. Hence, the maximum shear is obtained by rotating such that $2\theta = 90^o$, i.e., by 45^o from the principal axes. Moreover, the maximum shear is seen from the Mohr's circle to be just $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$, as we have already seen. The Mohr's circle shows that the direct stresses in this coordinate system are both equal to $\frac{1}{2}(\sigma_1 + \sigma_2)$.

Qu.: What point on the Mohr's circle represents the state of stress at a point?

It's a trick question. It's the ***whole*** Mohr's circle which represents the state of stress at a point.

Qu.: But then what do A and B represent?

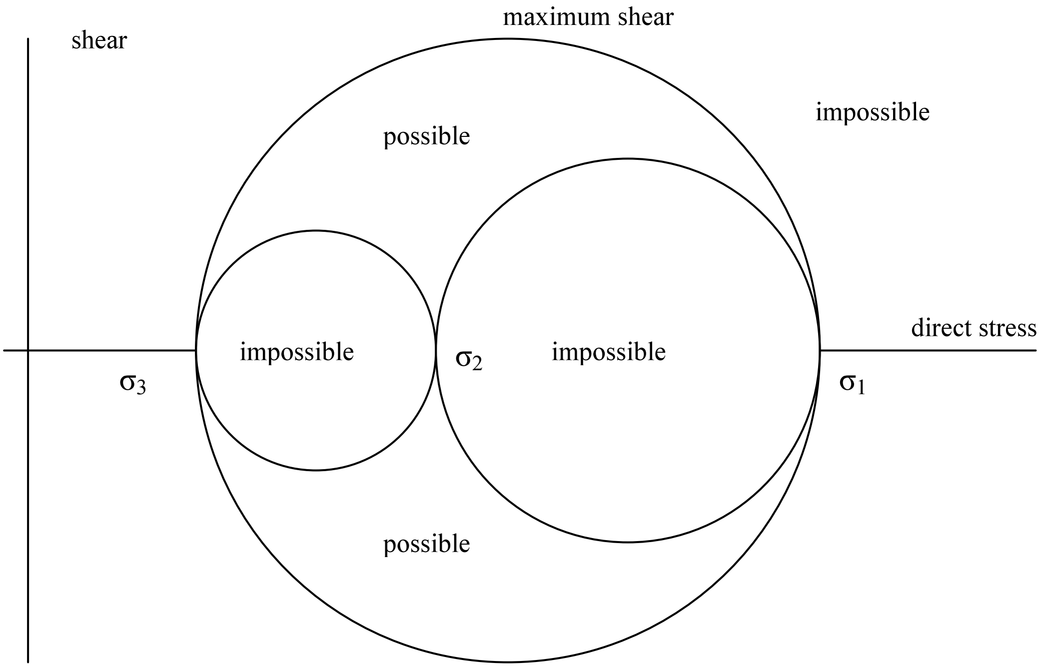
The diametrally opposite points A and B represent the components of the stress tensor with respect to a specific coordinate system, namely the coordinate system at an angle θ to the principal axes. The difference between the whole Mohr's circle and the points A,B is the same as the difference between the stress tensor and the stress matrix. The stress matrix gives the components of the tensor with respect to a given coordinate system. So,

Mohr's circle ≡ stress tensor

Points A,B ≡ stress matrix

Qu.: What are the Mohr's circles for a 3D state of stress?

Now there are three principal stresses, so we can draw three different Mohr's circles according to which pair of principal stresses we choose to define the circle. What we get is,



Again this shows that the maximum shear is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$.

A rotation of the coordinate system in any one of the three principal planes moves the stress components around the relevant circle.

But a general 3D rotation is not in one of the principal planes. For a general 3D rotation the stress components acting on an element of area are given by a point ***within*** the region between the inner and outer circles (marked "possible").

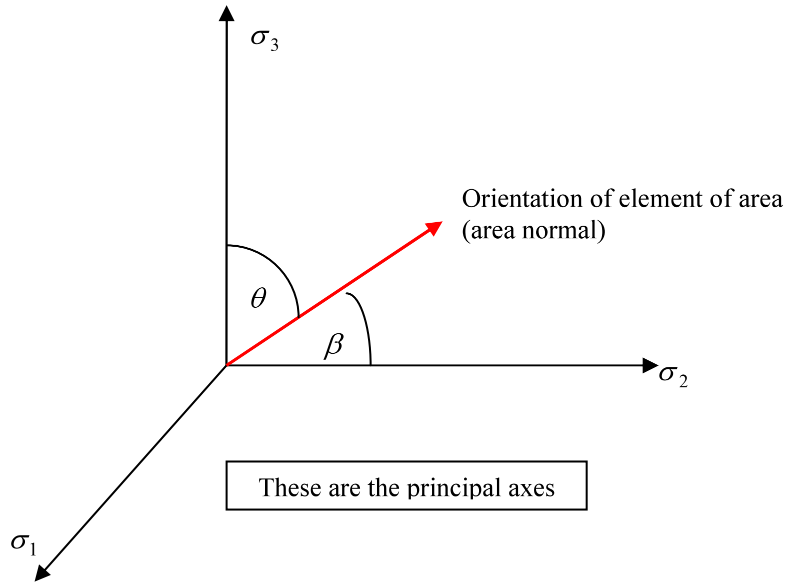
The points within the inner circles, or outside the outer circle, cannot be reached, i.e., there is no orientation of an area element such that the area element is acted upon by such a combination of direct and shear stresses.

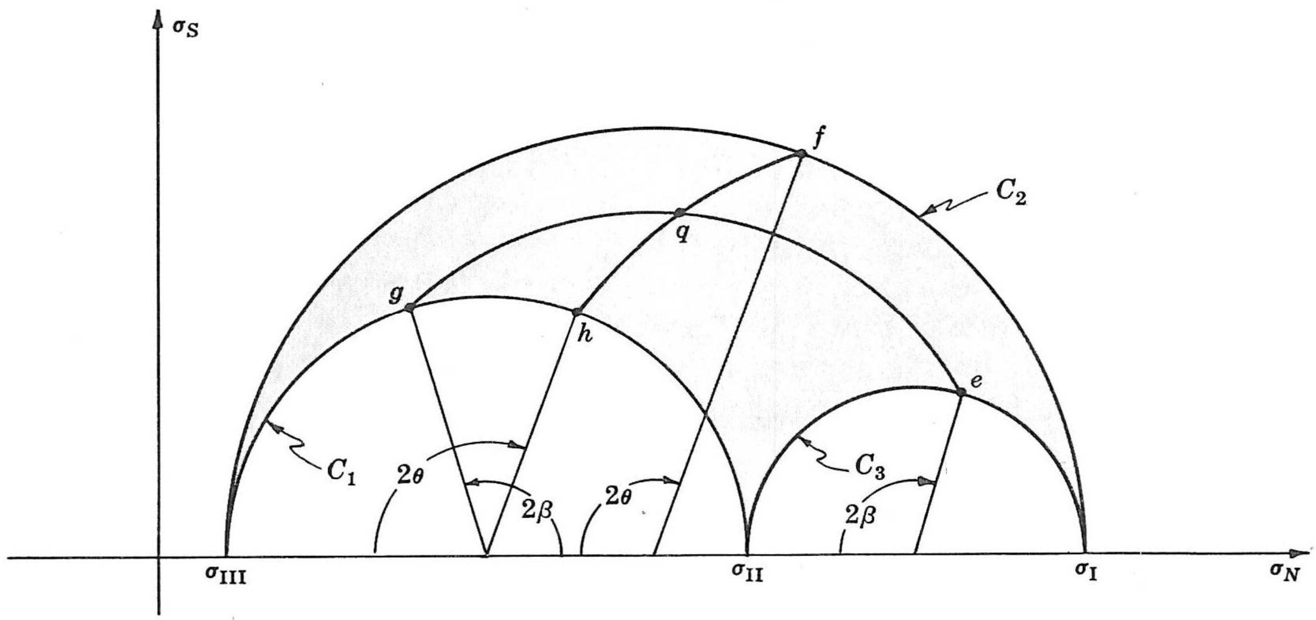
Qu.: What point on the 3D Mohr's circles represents the state of stress at a point?

Not falling for it a second time, eh? You think that the whole of the three circles represents the state of stress at the point? Good try, but no. It is the whole of the "possible" region between the inner and outer circles which represents the state of stress, i.e., the stress tensor. The correspondence in 3D is,

- The region between the inner & outer Mohr's circles ≡ stress tensor
- Any points in this region ≡ the direct and shear stresses on an element of area

Qu.: How do I find the point on the 3D Mohr's circles corresponding to a given orientation of the element of area?





Objects versus Components

Tensors, including vectors, are mathematical objects in their own right – independent of the coordinate system used to describe them in terms of components.

	Object	**Components**
Vector	Arrow	$(v_x \quad v_y \quad v_z)$
Stress Tensor	Mohr's Circles	$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$