SQEP Tutorial Session 4: T72S01
Relates to Knowledge & Skills 1.5, 1.6, 1.7
Last Update: 18/11/13

Basic formulation of 3D continuum stresses and strains; Definitions in terms of forces and displacements (small strains); Why the tensors are symmetric; Equilibrium condition, within a body and on a boundary; Isotropic elastic moduli; Derivation of G = E/2(1+ν); Change of volume, relation to ν.

In this session small strains are assumed, and hence geometry changes are neglected.

Qu.: What is stress?

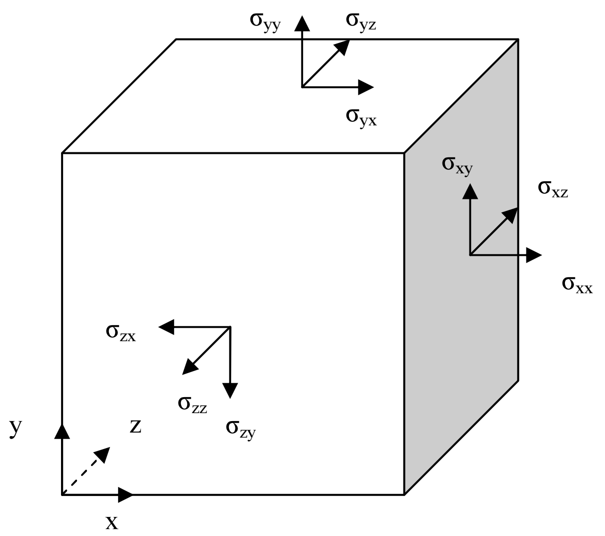
Roughly, stress is force divided by the area over which it acts.

Qu.: How many components of stress are there?

The force may be in any of the three coordinate directions. The orientation of the element of area is described by its normal vector, and hence can also be in any of the three coordinate directions. So there are 3 x 3 = 9 components of stress - ?

Qu.: Shouldn't that be six components of stress?

Yes. The nine components reduce to six because the stress matrix is symmetric, i.e., $\sigma_{xy} = \sigma_{yx}$, etc.



Note the directions of σ_{zx} and σ_{zy}, as a result of the front face of the cube having its normal in the negative z-direction.

The directions of all forces are reversed on the opposite faces.

Hence it can be seen from the diagram that,

σ_{xy} produces a moment which is in the opposite sense to that caused by σ_{yx};

σ_{xz} produces a moment which is in the opposite sense to that caused by σ_{zx};

σ_{yz} produces a moment which is in the opposite sense to that caused by σ_{zy};

Qu.: Why is the stress matrix symmetric?

For a cube of material, the net moment about the z-axis caused by σ_{xy} and σ_{yx} cancels to give zero if and only if $\sigma_{xy} = \sigma_{yx}$, and similarly for the other shear terms. But equilibrium requires there to be no net moment acting on the element of material, hence the stress matrix is symmetric.

Equilibrium of moments implies symmetry of the stress matrix

Note that this is only true if we assume there is no applied "body moment", i.e., a distributed applied moment per unit volume. In all practical engineering applications you are likely to come across this will be the case – unless possibly you are ever involved in NMR scanners. (Most atoms or nuclei tend to have a non-zero magnetic moment, and a magnetic moment which is not aligned with an applied magnetic field will experience a moment tending to make it align. So, if you are lying in an NMR scanner listening to it whir and bang you may wish to contemplate the unusual non-symmetric nature of your brain's stress tensor. However, in the circumstances, it may be another type of unusual stress which will be occupying your attention).

Qu.: What is meant by the diagonal and off-diagonal components?

The diagonal components are just the direct stresses (tensile or compressive), and the off-diagonal components are the shear stresses.

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \tag{1}$$

Qu.: What constraint does equilibrium of forces impose on the stress matrix?

If the stresses on the opposite faces are equal (and opposite), then they obviously all cancel and the net force is zero. However, this only applies if the stress is uniform over that region of the body. In general, any stress component will change slightly between one face and the opposite face. Consider all sources of force in the x-direction on the above element of material.

Firstly there are the x-forces arising from the stress component σ_{xx}. But this acts both on the face whose normal is parallel to x and also on the face whose normal is anti-parallel to x. Taking the latter to be located at the x-coordinate 'x', the former is therefore at x-coordinate $x + \delta x$, where δx is the size of the element in the x-direction. Hence, the net contribution to the x-force of the σ_{xx} stress is,

$$[\sigma_{xx}(x + \delta x) - \sigma_{xx}(x)]\delta y \delta z \tag{2}$$

where δy and δz are the sizes of the element of material in the y and z directions, and hence $\delta y \delta z$ is the area of the x-faces.

Next consider the x-forces arising from the shear component σ_{yx}. There is a positive contribution from the y-face at coordinate position $y + \delta y$, and a negative contribution from the y-face at coordinate position 'y'. Hence, the net contribution to the x-force of the σ_{yx} stress is,

$$[\sigma_{yx}(y + \delta y) - \sigma_{yx}(y)]\delta x \delta z \tag{3}$$

noting that $\delta x \delta z$ is the area of the y-faces. By symmetry we clearly get a net contribution to the x-force from the σ_{zx} shear stress of,

$$[\sigma_{zx}(z+\delta z)-\sigma_{zx}(z)]\delta x \delta y \tag{4}$$

But by the definition of the derivative of a function we can write,

$$[\sigma_{xx}(x+\delta x)-\sigma_{xx}(x)]=\frac{\partial \sigma_{xx}}{\partial x}\delta x \tag{5}$$

$$[\sigma_{yx}(y+\delta y)-\sigma_{yx}(y)]=\frac{\partial \sigma_{yx}}{\partial y}\delta y \tag{6}$$

$$[\sigma_{zx}(z+\delta z)-\sigma_{zx}(z)]=\frac{\partial \sigma_{zx}}{\partial z}\delta z \tag{7}$$

Hence, the total x-force acting on the element of material can be written simply as,

$$\delta F_x=\frac{\partial \sigma_{xx}}{\partial x}\delta x\delta y\delta z+\frac{\partial \sigma_{yx}}{\partial y}\delta y\delta x\delta z+\frac{\partial \sigma_{zx}}{\partial z}\delta z\delta x\delta y=\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\delta y\delta z \tag{8}$$

Hence, if there is no applied load acting on the material at this location then we conclude that equilibrium of the x-force requires,

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}=0 \tag{9}$$

Similarly, equilibrium of the y and z forces require,

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}=0 \tag{10}$$

and

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=0 \tag{11}$$

In a compact tensor notation all three equations may be written,

$$\sigma_{ij,i}=0 \tag{12}$$

(see Session 6 for explanation). Since the stress matrix is symmetric this can also be written,

$$\sigma_{ji,i}=0 \tag{13}$$

It there is an applied body force at the point in question, given by a vector $\overline{b}$ per unit volume, then equilibrium of forces becomes, more generally,

$$\sigma_{ji,i}=-b_j \tag{14}$$

So that the total force on the element is zero, i.e., $\sigma_{ji,i}+b_j=0$.

Equilibrium of forces implies $\sigma_{ji,i}=-b_j$, or $\sigma_{ji,i}=0$ for no applied body force

Qu.: What stress components must be zero at a free surface?

If the free surface is normal to x, then the stress components containing x must be zero, i.e., $\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0$. In general, the other three stress components may not be zero.

Qu.: What is strain?

Direct strain is the fractional change of length. More generally, to accommodate shear strains, strain is the difference in the displacement between two points divided by their distance apart. But there's a subtlety…

Qu.: Given the vector displacement $\bar{u}(\bar{r})$ at all points $\bar{r}$, what is the direct strain?

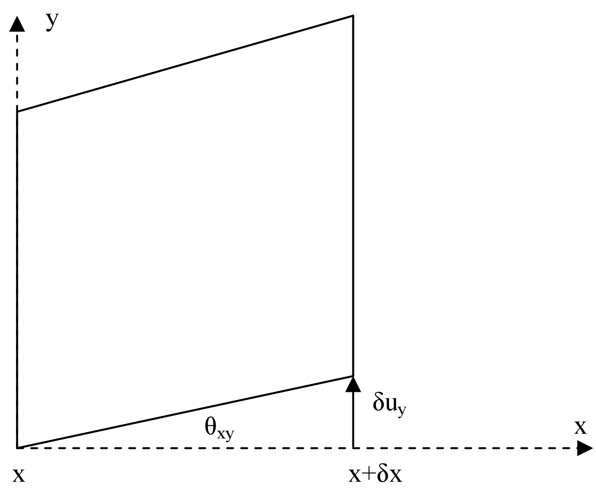
Start by considering the direct x-strain. Suppose two points are a distance δx apart in the x-direction, being located at x and $x+\delta x$ respectively. The x-displacement at these two points is $u_x(x)$ and $u_x(x+\delta x)$. Hence, their relative displacement (which equals the change in their distance apart in this case) is just $u_x(x+\delta x) - u_x(x)$. The x-strain is thus $\varepsilon_{xx} = \frac{u_x(x+\delta x) - u_x(x)}{\delta x} = \frac{\partial u_x}{\partial x}$. Similarly we get $\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$ and $\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$.

Qu.: Given the vector displacement $\bar{u}(\bar{r})$ at all points $\bar{r}$, what is the shear strain?

Suppose now that the same two points, at a distance δx apart in the x-direction, have differing displacements in the y-direction, namely $u_y(x)$ and $u_y(x+\delta x)$. The relative y-displacement is just $\delta u_y = u_y(x+\delta x) - u_y(x)$. Dividing by the distance the points are apart gives one measure of shear strain: $\gamma_{xy} = \frac{u_y(x+\delta x) - u_y(x)}{\delta x} = \frac{\partial u_y}{\partial x}$. This is the 'engineering shear strain'. *Note that this expression for the engineering shear strain is valid only if* $\frac{\partial u_x}{\partial y} = 0$*, as it is for the following illustration…*

Qu.: What is the geometrical interpretation of engineering shear strain?

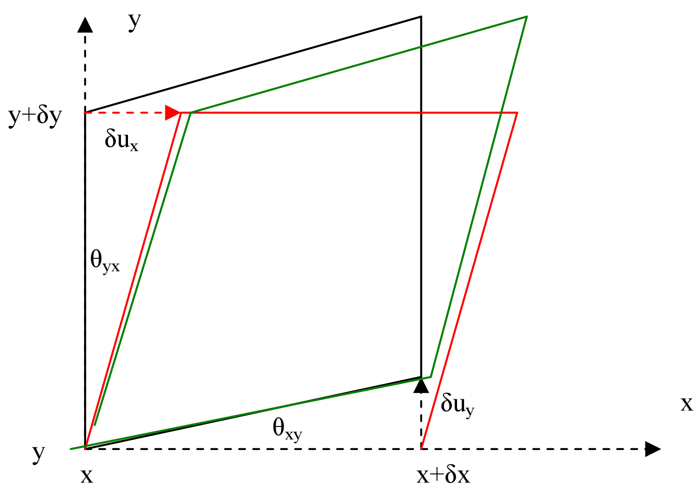
The above situation is illustrated by,



When the distortion is of this form, the engineering strain, γ_{xy}, is equal to the angle by which the element of material is distorted by shear, θ_{xy} (for small angles so that $\gamma_{xy} = \frac{\delta u_y}{\delta x} = \tan\theta_{xy} \approx \theta_{xy}$). *Note again that this expression is valid only for distortion as illustrated above so that* $\frac{\partial u_x}{\partial y} = 0$.

Qu.: Why is the strain matrix symmetric?

Consider the deformation due to a displacement in the x-direction which varies in the y-direction (i.e., the opposite way around from the above illustration). We get, in red,



For small strains the angles are given by $\theta_{xy} \approx \frac{\delta u_y}{\delta x}$ and $\theta_{yx} \approx \frac{\delta u_x}{\delta y}$. If we choose the magnitudes of the two shearing angles, θ_{xy} and θ_{yx}, to be the same, then the two (black and red) deformed shapes differ only by a rigid body rotation. But a rigid body rotation does not deform the shape of the element, and hence does not correspond to any strain. Consequently we must regard the black and red cases illustrated above as representing the same state of shear strain if $\theta_{xy} = \theta_{yx}$. Now suppose we impose both distortions together – producing the shape shown as green above. The engineering shear strain will be the sum of θ_{xy} and θ_{yx}, so the general expression for engineering shear strain is,

$$\gamma_{xy} = \gamma_{yx} = \theta_{xy} + \theta_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \qquad (15)$$

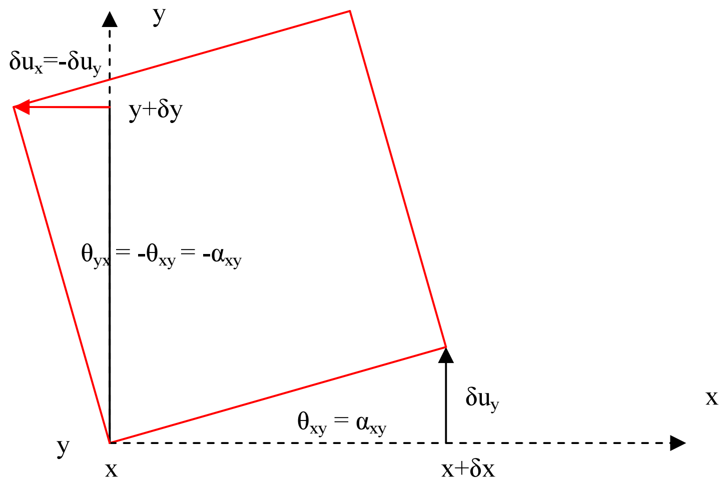
However, the tensorial shear strain is defined with a factor of a half, thus,

(Tensorial) Shear Strain:
$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\theta_{xy} + \theta_{yx}\right) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\gamma_{xy}}{2} \qquad (16)$$

When θ_{xy} and θ_{yx} are different, their difference equals the rigid body rotation.

Rotation:
$$\alpha_{xy} = \frac{1}{2}\left(\theta_{xy} - \theta_{yx}\right) \qquad (17)$$

Thus, if θ_{xy} and θ_{yx} are equal and opposite in sign then we get a pure rotation:-



In compact tensor notation the definition of all strain components is $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Qu.: Why is the factor of ½ needed in the definition of the strains?

The factor of ½ is needed so the definition reduces to the usual definition for direct (tensile) strains. We then need to retain the factor of ½ for the shear strains since otherwise the strain components would not form a tensor, i.e., they would not transform under rotations the way tensor components should (see Session 6).

Qu.: Do tensorial shear strains differ from engineering shear strains?

Yes. By that factor of ½ . We have,

Engineering shear strain, γ_{xy} = twice tensorial shear strain, $2\varepsilon_{xy}$

Qu.: For uniaxial stress, how are stress and strain related?

Hooke's Law: $\sigma_x = E\varepsilon_x$. For a general material, the value of Young's modulus might differ in different directions. For an isotropic material, E is the same in all directions.

Qu.: What is the Poisson effect?

When a uniaxial stress is applied, say σ_x , as well as the x-strain there are also strains in the y and z directions equal to a fraction ν of the x-strain, but in the opposite sense. Hence the three strains resulting from a uniaxial x-stress are $\varepsilon_x = \sigma_x / E$, $\varepsilon_y = -\nu\sigma_x / E$ and $\varepsilon_z = -\nu\sigma_x / E$.

Qu.: What is the 3D isotropic elastic relationship between stress and strain?

This follows from the above Poisson effect expressions by linearly superimposing three uniaxial cases, thus,

$$
\begin{aligned}
E\varepsilon_x &= \sigma_x - \nu\sigma_y - \nu\sigma_z \\
E\varepsilon_y &= -\nu\sigma_x + \sigma_y - \nu\sigma_z \\
E\varepsilon_z &= -\nu\sigma_x - \nu\sigma_y + \sigma_z
\end{aligned}
\qquad (18)
$$

Or, in matrix notation,

$$E\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{pmatrix}=\begin{pmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} \quad (19)$$

Qu.: What about the relationship between shear stress and strain?

A shear stress only causes shear deformation in the same sense, so we have simply,

$$2G\begin{pmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{pmatrix}=G\begin{pmatrix}\gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{pmatrix}=\begin{pmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{pmatrix} \quad (20)$$

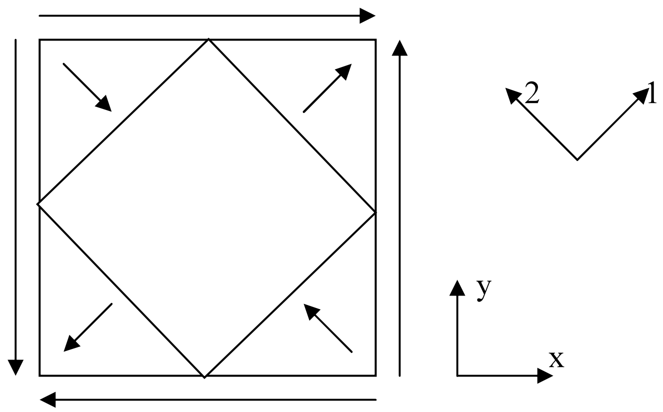
where G is the shear modulus.

Qu.: Is the shear modulus independent of E and ν?

No. It could not be. This is because shear stresses are really just an artefact of the coordinate system employed. If we rotate to the principal coordinate system then all shear stresses disappear. So E and ν must suffice, and G must be expressible in terms of them.

Qu.: So what does pure shear stress look like in principal coordinates?

If we have a pure shear stress of τ, then the principal coordinates are at 45° with $\sigma_1=\tau$ and $\sigma_2=-\tau$. This can be deduced from the diagram below,



The resolved force on each small triangle is $2\times(\tau/2))\times\left(1/\sqrt{2}\right)=\tau/\sqrt{2}$. But the length of the side of the smaller, rotated, square is only $1/\sqrt{2}$, so the direct stresses are just equal to τ in magnitude, one tensile and one compressive, as illustrated.

Qu.: So what does pure shear strain look like in principal coordinates?

Because strain is also tensorial, the same relationship must hold, i.e., we have principal strains $\varepsilon_1=\varepsilon_{xy}$ and $\varepsilon_2=-\varepsilon_{xy}$.

Qu.: What is G in terms of E and ν?

In the original coordinate system we have $2G\varepsilon_{xy}=\sigma_{xy}=\tau$. In the principal coordinate system we have $E\varepsilon_1=\sigma_1-\nu\sigma_2=\tau-\nu(-\tau)=(1+\nu)\tau$. But since $\varepsilon_1=\varepsilon_{xy}$ we therefore have, $\varepsilon_1=(1+\nu)\tau/E=\varepsilon_{xy}=\tau/2G$. Hence, $(1+\nu)/E=1/2G$, or,

$$G = \frac{E}{2(1+\nu)} \tag{21}$$

Qu.: What is the volumetric strain?

Shear strains do not change the volume. The sides of a unit cube are changed by the direct strains by a factor of $1+\varepsilon_x$, $1+\varepsilon_y$ and $1+\varepsilon_z$. Hence, the change of volume of a unit cube is just $(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)-1 \approx \varepsilon_x + \varepsilon_y + \varepsilon_z$. The latter expression results from retaining only the leading order (linear) terms in the expansion of the brackets. This is the volumetric strain, i.e., the fractional change in volume, in the small strain approximation,

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{22}$$

Qu.: How does the volumetric strain in isotropic elasticity relate to Poisson's ratio?

From the 3D Hooke's Law it follows that,

$$\frac{\delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{23}$$

Consequently, in elasticity, the volume expands if and only if the hydrostatic (mean) stress, $\sigma_H = (\sigma_x + \sigma_y + \sigma_z)/3$, is positive, and shrinks if it is negative – as one would expect. But this is only because $\nu < \frac{1}{2}$. In contrast, a material with $\nu = \frac{1}{2}$ does not change volume, irrespective of the stress. Thus,

$\nu = \frac{1}{2}$ is the signature of incompressibility.