SQEP Tutorial Session 3: T72S01
Relates to Knowledge & Skills 1.3

Beam Theory 2: Derivation of displacements and rotations; Solution of statically indeterminate problems

Last Update: 22/10/13

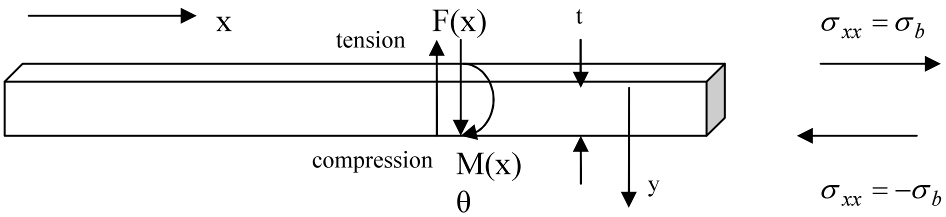
Warning: My sign convention may differ from text books.

Qu.: If a beam is encastre at both ends, do the equations $\frac{dF}{dx}=\lambda$ and $\frac{dM}{dx}=F$ suffice to solve for the moment diagram?

No. The reason is simply that the boundary conditions are that the ends do not rotate, i.e., $\theta=0$ at $x=0$ and $x=L$. But there is no way of imposing these boundary conditions so far, since θ does not yet appear in our set of equations.

Qu.: What variables describe the deformation of a beam?

The appropriate variables are the vertical displacement, y, and the rotation, θ. My convention is that the y displacement is measured downwards, and hence, because beams generally deflect downwards, y will most often be positive. The rotation, θ, is measured clockwise.



Qu.: How do we relate these deformation variables to the load resultants?

The beam becomes curved as a response to the bending moment. So the curvature of the beam must be proportional to the bending moment (since we are restricting attention to linear elasticity). The curvature is the reciprocal of the radius of curvature, $1/\rho$, so we expect $(1/\rho)\propto M$. (In session 1 ρ was called R).

Qu.: What is the radius of curvature of the beam?

(Consult the diagram below). For small angles we can approximate $\cos\theta\approx 1$, so that there is negligible difference between the arc-length of the beam between the positions indicated and δx. Consequently, $\rho\delta\theta\approx\delta x$, and the curvature of the beam is just,

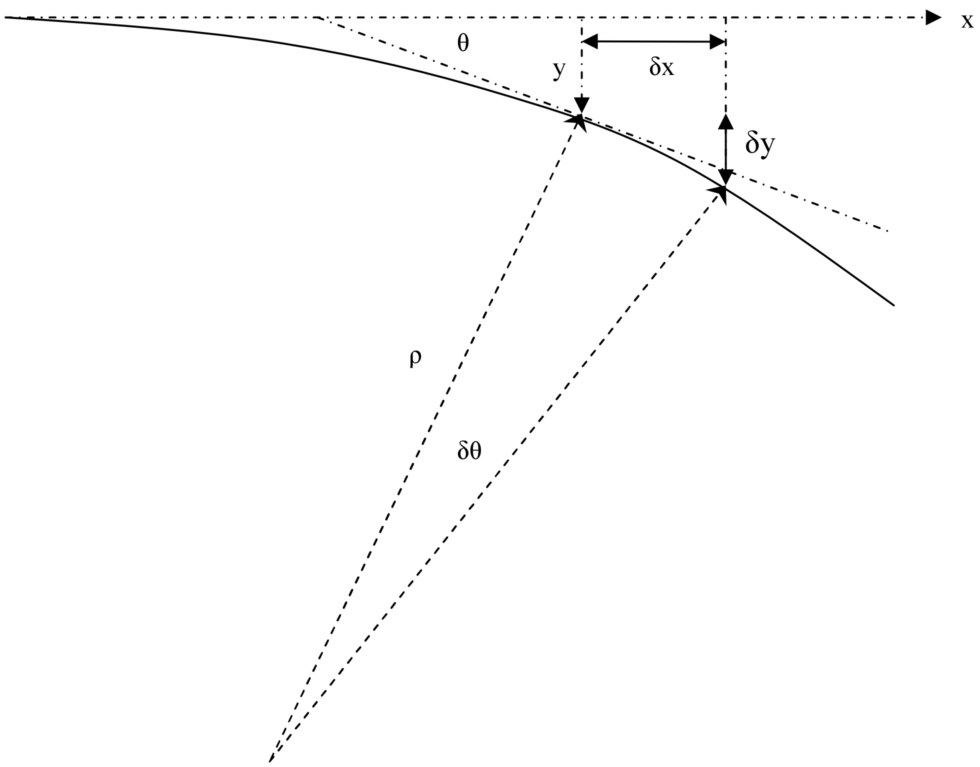
$$\frac{1}{\rho}=\frac{d\theta}{dx} \qquad (1)$$

Qu.: For a given M, what else does the curvature of the beam depend upon?

The curvature also depends on the material's Young's modulus and the beam's second moment of area (I), both of which affect the stiffness and hence the curvature for a given applied moment.

Qu.: If E is increased, what happens to the curvature?

The curvature decreases because the beam gets stiffer, and must do so proportionately. So curvature $\propto 1/E$.



Qu.: If I is increased, what happens to the curvature?

The curvature decreases because the beam gets stiffer, and must do so proportionately. So curvature $\propto 1/I$.

Qu.: So what is the expression for curvature?

From the above observations we conclude that $\frac{1}{\rho} = \frac{d\theta}{dx} \propto \frac{M}{EI}$. The constant of proportionality must be unity because…

In session 1 we derived two expressions for the outer fibre bending stress, one interms of the moment, $\sigma_b = \frac{Mb}{I}$, and one in terms of the strain, which is simply $\frac{b}{\rho}$ and hence $\sigma_b = E\frac{b}{\rho}$. Equating these two expressions for the stress gives,

$$\frac{1}{\rho} = \frac{M}{EI} \tag{2}$$

Qu.: How does y relate to θ?

The displaced shape of the beam is just y plotted against x. The gradient of this curve at any point is $\frac{dy}{dx}$. But the gradient of a curve is the tan of the angle made by its tangent, i.e., $\tan\theta$. Assuming we are always dealing with very small beam displacements, in comparison with the beam length, the angle, θ, is small and we may approximate $\tan\theta \approx \theta$. This provides our fourth and last beam equation,

$$\theta = \frac{dy}{dx} \tag{3}$$

Qu.: What is the full set of beam equations?

We now have all four beam equations. They are,

$$\frac{dF}{dx} = \lambda; \qquad \frac{dM}{dx} = F; \qquad \frac{d\theta}{dx} = \frac{M}{EI}; \qquad \theta = \frac{dy}{dx} \tag{4}$$

Qu.: Is there any other way of expressing beam theory?

Well, we can always replace the four first order equations with one fourth order equation. We have,

$$\frac{d^4 y}{dx^4} = \frac{d^3\theta}{dx^3} = \frac{1}{EI}\frac{d^2 M}{dx^2} = \frac{1}{EI}\frac{dF}{dx} = \frac{\lambda}{EI} \tag{5}$$

But it is generally more transparent to deal directly with the four first order equations.

Qu.: What is a "statically indeterminate" problem?

A problem is said to be "statically indeterminate" if it cannot be solved using the equations which result from equilibrium only, i.e., the two equations $\frac{dF}{dx} = \lambda$ and $\frac{dM}{dx} = F$.

Qu.: With our two extra equations can we now solve statically indeterminate problems?

Yes. Here's an example…

<u>Beam encastre at both ends with a point load in the middle</u>

Call the point load W. It suffices to consider only the left half of the beam,

$x < L/2$: $\quad F = -W/2 \quad$ and $\quad \frac{dM}{dx} = F \quad$ gives $\quad M = -Wx/2 + M_0$

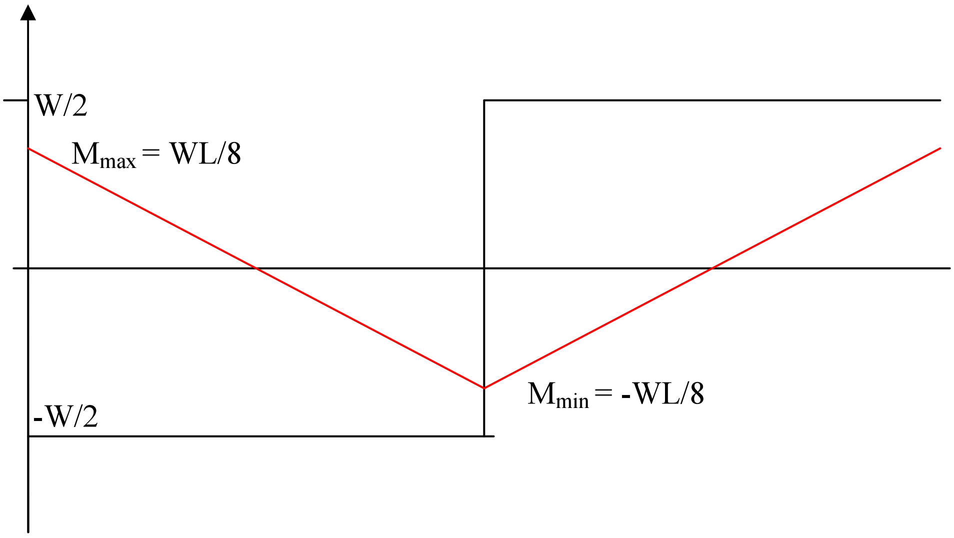
The bending moment at the end, M_0, is unknown. Integrating $\frac{d\theta}{dx} = \frac{M}{EI}$ gives,

$x < L/2$: $\quad EI\theta = -Wx^2/4 + M_0 x$

where the integration constant is zero because $\theta = 0$ at $x = 0$. Symmetry requires the rotation at the beam centre be zero, so that,

$$-\frac{W}{4}\left(\frac{L}{2}\right)^2 + M_0\frac{L}{2} = 0 \qquad \text{and hence,} \qquad M_0 = \frac{WL}{8}$$

The bending moment diagram is shown below.



Note that the maximum bending moment is half that for a simply supported beam with a central point load. Making the beam encastre at both ends causes it to be twice as strong (at least, as judged from the maximum elastic stress).

Summary

The most important things to remember are the four beam equations,

$$\frac{dF}{dx} = \lambda\,; \qquad \frac{dM}{dx} = F\,; \qquad \frac{d\theta}{dx} = \frac{M}{EI}\,; \qquad \theta = \frac{dy}{dx}$$