SQEP Tutorial Session 2: T72S01
Relates to Knowledge & Skills 1.1 & 1.2
Last Update: 25/10/13
Beam Theory 1: Derivation of shearing force and bending moment diagrams; Solution of statically determinate beam problems

λ = load per unit length (units N/m); M = bending moment (units Nm);
Warning: My sign convention may differ from text books.

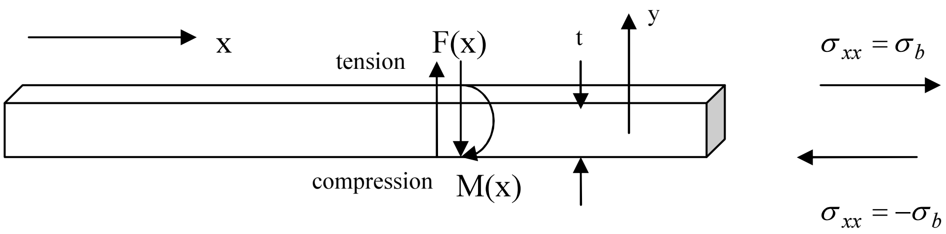
Qu.: What is beam theory?

Beam theory is an approximation to the full continuum treatment of the elasticity problem. It consists of approximating the six components of stress to just two: one shear and one direct stress. In addition, the direct stress is assumed to have the form of a bending stress, i.e. to vary linearly across the section of the structure and to correspond to no net force. In beam theory the severity of the loading across any section of the structure is described by just two 'load resultants': the shearing force, F, and the bending moment, M.

Qu.: When is beam theory a reasonable approximation?

I will not attempt a rigorous derivation. If you do, the key idea is that "plane sections remain plane", i.e. that any plane cross-section of the structure is still planar after loading. In practice, for beam theory to be reasonably accurate the aspect ratio of the structure should be sufficiently large, i.e. the length to depth ratio should be greater than perhaps ~5, and preferably >10. However, beam theory estimates can be very useful even for small aspect ratios – as long as you remember that they won't necessarily be very precise. Also, beam theory does not capture local stresses. Beam theory only describes the linearised stresses over a cross section. So the stresses local to a concentrated load will not be reproduced. Nor will the detailed stress distributions around realistic support arrangements, e.g., near welded or bolted connections, etc.

Qu.: What are the conventional axes for beam theory?



Qu.: What is the 'shearing force', F(x)?

It is the total shear force integrated over the cross section at a given axial position, x. The shear in question is σ_{xy}. The direction of F is upwards on the material to the left, and downwards on the material to the right. (That's the convention I'll use – it may not be universal). This means that $F = \int \sigma_{xy} dA$, where $dA = dydz$.

Qu.: What is the 'bending moment', M(x)?

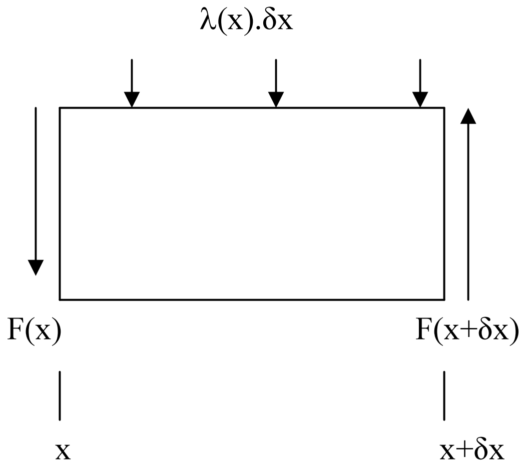
This is just the usual bending moment acting at axial position x. Its sense is defined such that tension on the top surface is positive (again, my convention). Hence, the bending stress at position 'x' is $\sigma_b = \frac{My_{\max}}{I}$. ($y_{\max}$ was called b in session 1).

Qu.: How is a beam loaded?

A beam is taken to be loaded by a load per unit length, λ (units: N/m), acting perpendicularly to the beam length.

Qu.: How is the shearing force related to the applied load?

Equilibrium of forces provides a simple relation,

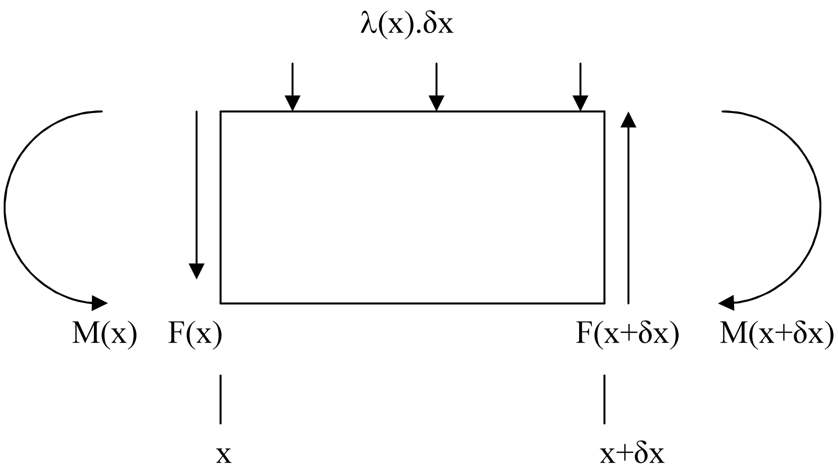


i.e., $F(x+\delta x) - F(x) = \lambda(x).\delta x$. Hence, $\frac{dF}{dx} = \lambda$. (1)

NB: Recall that the definition of the derivative is $\frac{dF}{dx} \equiv LIM\left(\frac{F(x+\delta x) - F(x)}{\delta x}\right)_{\delta x \to 0}$ (2)

Qu.: How is the bending moment related to the shearing force?

Equilibrium of moments provides a simple relation,



So, $M(x+\delta x)-M(x)=F(x).\delta x$. Hence, $\dfrac{dM}{dx}=F$. (3)

Qu.: We know that the bending stress corresponding to M varies linearly across the section, but how is the shear stress corresponding to F distributed?

In general the distribution can be quite complicated (though there is a simple formula, see below). What we do know is that the xy shear stress must be zero at the top and bottom surfaces of the beam (assuming there is no applied shear load). We also know that, at locations where F is non-zero, the shear stresses is non-zero at some points across the section because $F=\int \sigma_{xy} dA$. The simplest distribution is therefore parabolic, and this is correct for a beam of rectangular section, giving,

$$\sigma_{xy}=\frac{F}{tw}\left(\frac{3}{2}-6\frac{y^2}{t^2}\right) \tag{4}$$

where t is the depth of the beam and w is the beam width, and y is measured from the centre of the section. Hence, for a rectangular section the maximum shear is $^3/_2$ times the mean shear, at $y = 0$.

For other sections, the shear distribution is more involved and is given by,

$$\tau=\frac{FA\bar{y}}{w(y)I} \tag{5}$$

This gives the shear stress at a distance y from the neutral axis at a section where the total shearing force is F. This applies to an arbitrary shape of section, and $w(y)$ is the width of the section at the distance y from the neutral axis. Finally, the quantity $A\bar{y}$ is defined by,

$$A\bar{y}=\int_{y}^{y_{MAX}} yw(y).dy \equiv \int_{y}^{y_{MAX}} ydA \tag{6}$$

Note that this integral is over the range y to $y_{\max}$, and hence becomes zero as y approaches the outer fibre *on either the tensile or the compressive side.* Hence the shear is zero at the outer fibre, as it should be. This formula can be used, for example, to show that for a solid circular section, the peak shear stress is 4/3 times the mean shear (and is maximum on the neutral axis).

Qu.: Is it simple to derive $\sigma_{xy}=\dfrac{F}{tw}\left(\dfrac{3}{2}-6\dfrac{y^2}{t^2}\right)$ for a rectangular section?

Yes. You can use the above formula. Alternatively, the equilibrium equation is $\dfrac{d\sigma_{xx}}{dx}+\dfrac{d\sigma_{xy}}{dy}=0$. But $\sigma_{xx}=\dfrac{2y}{t}\sigma_b=\dfrac{2y}{t}\cdot\dfrac{6M}{wt^2}$ so we have, $\dfrac{d\sigma_{xx}}{dx}=\dfrac{12y}{wt^3}\cdot\dfrac{dM}{dx}=\dfrac{12y}{wt^3}F$.

Hence, $\dfrac{d\sigma_{xy}}{dy}=-\dfrac{12y}{wt^3}F$, so $\sigma_{xy}=-\dfrac{6y^2}{wt^3}F+\text{constant}$. The value of the constant follows from the requirement that $\sigma_{xy}=0$ on the surfaces $y=\pm t/2$, which leads to

$\sigma_{xy}=\dfrac{F}{tw}\left(\dfrac{3}{2}-6\dfrac{y^2}{t^2}\right)$. QED.

Qu.: What does "simply supported" mean?

It means that the support provides a vertical load, which reacts the applied weight or load from the beam, but does not provide any restraint to rotation. This point of the beam is free to rotate.

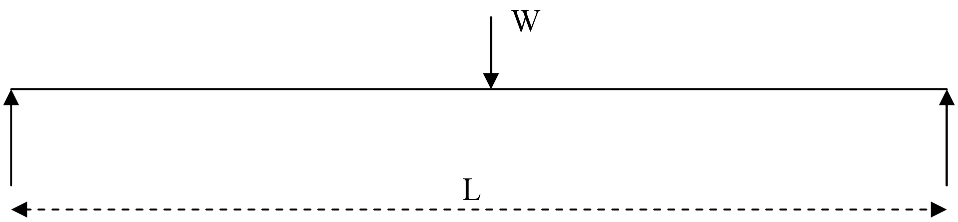
Qu.: What does "pin jointed" mean?

A pin joint is also free to rotate but can react load in any direction. The moment at a pin joint is zero because it does not resist rotation. A pin joint could develop a horizontal reaction (i.e., in the x-direction) as well as a vertical reaction (y-direction). But in simple beam theory we generally do not consider longitudinal loads (though they are obviously very important in frames and trusses).

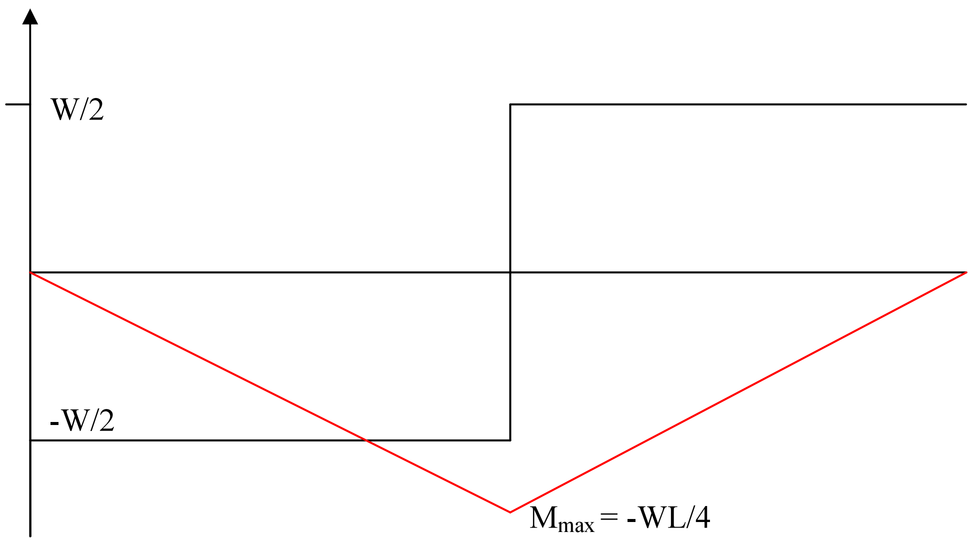
Qu.: What does "encastre" mean?

It means a support where the beam is built into a rigid wall, so that all movement, including rotation, is fully constrained.

Qu.: What are the shearing force and bending moment diagrams for a simply supported beam with a point load, W, at the centre?



By symmetry the end reactions are equal to W/2. Hence,



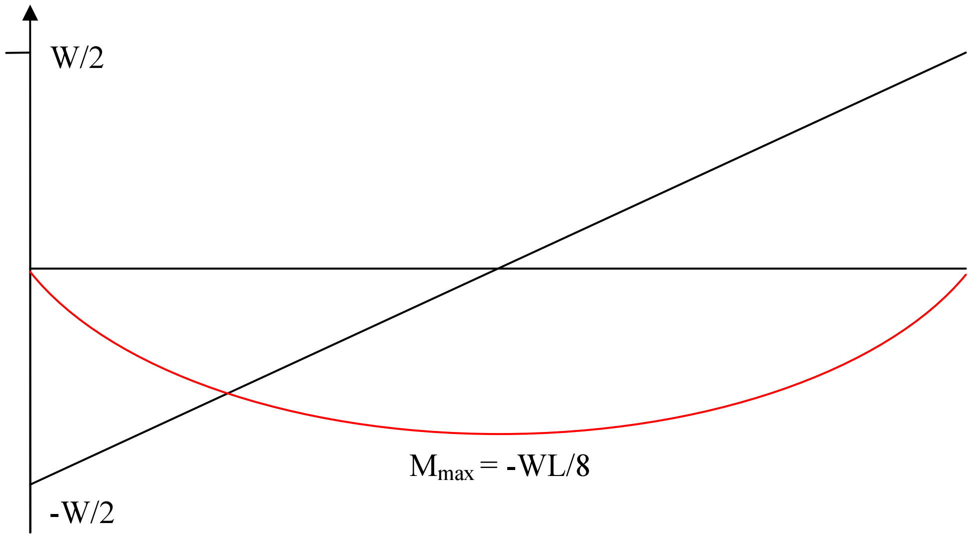
The bending moment distribution follows from $\frac{dM}{dx} = F$ by integrating. Considering the left half of the beam gives, $\frac{dM}{dx} = -\frac{W}{2}$, hence, $M = -\frac{W}{2}x + C$. The simple support at x = 0 means that M = 0 at x = 0, so that C = 0. Hence, $M = -\frac{W}{2}x$, for $x \le L/2$.

Hence, the maximum magnitude of moment is for $x = L/2$ and equals $M = -\frac{WL}{4}$.

The right-hand half of the beam has $\frac{dM}{dx} = F = +\frac{W}{2}$ which integrates to give $M = \frac{W}{2}x + C'$. The simple support at x = L means that M = 0 at x = L, so that $C' = -\frac{W}{2}L$. Hence, $M = -\frac{W}{2}(L - x)$.

Qu.: What are the shearing force and bending moment diagrams for a simply supported beam with a uniformly distributed load, W?

The ends reactions are still W/2 by symmetry. But the shearing force now decreases linearly due to the uniform loading: $F = W\left(\frac{x}{L} - \frac{1}{2}\right)$. You can derive this by integrating the first beam equation $\frac{dF}{dx} = \lambda$ where the load per unit length is $\lambda = W/L$.



The bending moment distribution follows from $\frac{dM}{dx} = F$ by integrating. Hence, $\frac{dM}{dx} = W\left(\frac{x}{L} - \frac{1}{2}\right)$ implies $M = W\left(\frac{x^2}{2L} - \frac{x}{2}\right) + C$. The simple support at x = 0 means that M = 0 at x = 0, so that C = 0. Hence, $M = W\left(\frac{x^2}{2L} - \frac{x}{2}\right)$. Hence, the maximum magnitude of moment is for $x = L/2$ and equals $M = -\frac{WL}{8}$.

In summary, the statically determinate beam equations are $\frac{dF}{dx} = \lambda$ and $\frac{dM}{dx} = F$