## T72S01 Session 12 – Thermal Stresses (Linear Elasticity Only)

Last Update: 18/5/14

*Thermal stresses: formulation of the general elastic problem; Constrained and unconstrained cases; Uniform temperature gradient gives zero stress if unconstrained; Examples, e.g. temperature gradient through a vessel wall; Anisotropy and inhomogeneity of α; Basics of heat transfer: Conduction equation; Convective boundary conditions, heat transfer coefficient; 1D temperature distribution, transient versus steady state*

Qu.:    Revision- What are the equations for the general elasticity problem?

When there are no thermal stresses, the three sets of equations are (as discussed in session 7),

Equilibrium: $\qquad\qquad\qquad\quad \sigma_{ij,j} = -b_i$ $\qquad\qquad\qquad\qquad\qquad$ (1)

Hooke's Law: $\qquad\qquad\qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ $\quad$ (may be anisotropic) $\qquad$ (2)

Compatibility: $\qquad\qquad\quad \varepsilon_{i,j} = \dfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ $\qquad\qquad\qquad\qquad$ (3a)

The last, the compatibility equations, can alternatively be written without reference to the displacements, i.e., as,

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy} \;;\; \varepsilon_{xy,xz} + \varepsilon_{xz,xy} = \varepsilon_{yz,xx} + \varepsilon_{xx,yz} \;;\; \text{etc.} \qquad (3b)$$

In this formulation, $\bar{b}$ is the applied load per unit volume, and there will also be, in general, stresses and/or displacements prescribed as boundary conditions on the surface of the body.

Qu.:    How does "thermal loading" arise?

The reason why temperature distributions cause stress is that they cause strain. That is, materials generally expand when their temperature increases. The effect of this thermal expansion is to cause a purely thermal strain, $\varepsilon_{ij}^{th}$, on an element of material which is raised in temperature uniformly by $\Delta T$.

Qu.:    What is the "coefficient of thermal expansion" – and is it a scalar?

For small changes in temperature, the thermal strain will be proportional to the temperature change. The factor of proportionality is the coefficient of thermal expansion. Since strain is a 2nd rank tensor and temperature is a scalar, the most general "coefficient of thermal expansion" will be a 2nd rank tensor, $\alpha_{ij}$,

$$\varepsilon_{ij}^{th} = \alpha_{ij}\Delta T \qquad\qquad\qquad\qquad (4)$$

In general, for anisotropic materials, the expansion may be different in the three coordinate directions *and* thermal expansion might also cause shear strains.

However, an isotropic material will expand isotropically, in which case the coefficient of thermal expansion reduces to a scalar,

$$\varepsilon_{ij}^{th} = \alpha\Delta T\delta_{ij} \qquad\qquad\qquad\qquad (5)$$

Here $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$). For the examples below we shall assume isotropy, i.e., Equ.(5). There is a brief discussion of anisotropy later.

Qu.: Does thermal strain really vary linearly with temperature?

No. But this is addressed simply by accepting that the coefficient of expansion is temperature dependent. The temperature dependence is not dramatic. For example, a CMn steel might have $\alpha \sim 12 \times 10^{-6} /^o C$ at $20^oC$, increasing to $\alpha \sim 14 \times 10^{-6} /^o C$ at $350^oC$ (see R66).

Qu.: What are "instantaneous" and "average" coefficients of expansion?

The instantaneous coefficient of expansion applies over a small range of temperature around the temperature, T, at which it is specified. It is defined by,

$$\alpha^{instant}(T) = \frac{d\varepsilon^{th}}{dT} \Leftrightarrow d\varepsilon^{th} = \alpha^{instant}(T)dT \tag{6}$$

However, we often want to know the total thermal strain due to a large temperature change, from $T_1$ to $T_2$. In the case where the lower temperature is $20^oC$, this defines the average coefficient of expansion,

$$\varepsilon^{th} = \alpha^{average}(T_2)(T_2 - 20^o C) \tag{7}$$

From this it follows that the thermal strain increment due to a large temperature change from $T_1$ to $T_2$ is,

$$\Delta\varepsilon^{th} = \alpha^{average}(T_2)(T_2 - 20^o C) - \alpha^{average}(T_1)(T_1 - 20^o C) \tag{8}$$

The same thermal strain increment can be found in terms of the instantaneous coefficient of expansion, but this must be known at all temperatures from $T_1$ to $T_2$ and requires an integral to be carried out,

$$\Delta\varepsilon^{th} = \int_{T_1}^{T_2} \alpha^{instant}(T)dT \tag{9}$$

R66 most often quotes average coefficients of expansion. However, R66 does give instantaneous values as well for some materials. Care is required when taking data from other sources, since they sometimes omit to tell you which definition is being used.

Qu.: So, to find the stress we just plug the thermal strain into Hooke's Law…?

WRONG!

Qu.: Why? What stress arises if a body is changed in temperature uniformly, and the body is not externally restrained from expansion?

Assuming the material is homogeneous, this produces no stress. Why? Because the body is simply free to expand. But the body *does* expand, and so there is a thermal strain – namely the strain given by Equs.(4) or (5). There is strain, but no stress.

This demonstrates the most important fact…

Thermal strain does not, of itself, give rise to stress

Yes, of course. Here is one way in which thermal stresses can arise. Heat a bar to a higher temperature, but ensure that it cannot expand by restraining its ends. Since the expansion is now zero, the total (longitudinal) strain is zero. But we know that the free thermal strain would have been $\alpha\Delta T$ if the bar had not been restrained. So the restraint has caused an additional, equal and opposite, strain of $\varepsilon^{el} = -\alpha\Delta T$ to arise. This is the 'elastic' strain which corresponds to stress via Hooke's Law. The thermal stress is thus $\sigma = E\varepsilon^{el} = -E\alpha\Delta T$, compressive because the expansion has been prevented.

The total strain is the sum of the thermal and elastic strains, $\varepsilon^{total} = \varepsilon^{el} + \varepsilon^{th}$. For a case of complete restraint of the thermal expansion, like this example, the total strain is zero.

> It is only the elastic part of the strain which relates to stress via Hooke's Law

> It is the *total* strain which is observable as the change in the dimensions of the body. It is the *total* strain which is related to the displacements in the usual manner.

Qu.:    How is the general elastic thermal stress problem formulated?

The stresses must still be in equilibrium, whatever the nature of the loading, so Equ.(1) still holds.

Similarly, Hooke's Law, Equ.(2), still holds provided that by "strain" we mean the elastic strain, not the total strain or the thermal strain.

The most significant change is in regard to compatibility. It is the *total* strain which must be compatible with the displacements of the material.

Finally, we also have an extra equation – namely Equ.(4) or (5) which specifies the thermal strain.

The complete set of equations for a thermal stress problem is thus,

| | | |
|---|---|---|
| Equilibrium: | $\sigma_{ij,j} = -b_i$ | (10) |
| Hooke's Law: | $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^{el}$   (may be anisotropic) | (11) |
| Compatibility: | $\varepsilon_{i,j}^{total} = \dfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ | (12) |
| Thermal Expansion: | $\varepsilon_{ij}^{th} = \alpha_{ij}\Delta T$  or  $\varepsilon_{ij}^{th} = \alpha\Delta T\delta_{ij}$ | (13) |
| Plus: | $\varepsilon_{ij}^{total} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{th}$ | (14) |

Note that other loads may be applied as well as thermal loading. The compatibility equations, (12), may also be expressed in the alternative form $\varepsilon_{xx,yy}^{total} + \varepsilon_{yy,xx}^{total} = 2\varepsilon_{xy,xy}^{total}$, $\varepsilon_{xy,xz}^{total} + \varepsilon_{xz,xy}^{total} = \varepsilon_{yz,xx}^{total} + \varepsilon_{xx,yz}^{total}$, etc.

WRONG!

A body which is completely free can also give rise to thermal stress – but the thermal strains must be non-uniform. Thermal stresses can therefore arise in isotropic materials due to non-uniform temperature distributions. Alternatively, a uniform temperature change can result in thermal stresses if the coefficient of thermal expansion varies across the body (i.e., the body is inhomogeneous). Both of these situations cause non-uniform thermal strains, and hence thermal stresses (in general).

But not all non-uniform temperature distributions will give non-zero thermal stresses.

**Qu.:    But doesn't a temperature gradient cause thermal stress?**

Not necessarily, no. This is a common, but serious, misunderstanding.

It is quite common for analysts to assume that the thermal stresses in a structure are proportional to the temperature gradient in some region of the body. Actually, this can sometimes be a reasonable approximation – if the temperature gradient exists only over a portion of the body, and separates two regions of uniform, but different, temperature.

But if the body is unrestrained and has a temperature gradient which is uniform across the whole body, and assuming a homogeneous material, then the thermal stresses are…zero!

> If a body of homogeneous material is unrestrained and its temperature varies linearly with the Cartesian x,y,z coordinates, then the thermal stresses are zero.

**Qu.:    Is this simple to prove?**

Yes. Or, at least, it is simple so long as we take it on trust that the solution to the set of equations (10-14) is unique.

Assume that the stresses are indeed zero. This means that the elastic strains are zero, and hence that the total strains equal the thermal strains, i.e., $\varepsilon_{ij}^{total} = \alpha_{ij}\Delta T$. So we have already ensured that equations (10), (11), (13) and (14) are obeyed. The only thing left is compatibility. Now if we use the alternative form of the compatibility relations, i.e., $\varepsilon_{xx,yy}^{total} + \varepsilon_{yy,xx}^{total} = 2\varepsilon_{xy,xy}^{total}$, $\varepsilon_{xy,xz}^{total} + \varepsilon_{xz,xy}^{total} = \varepsilon_{yz,xx}^{total} + \varepsilon_{xx,yz}^{total}$, etc., we see that all terms are second order derivatives in the Cartesian coordinates. Since, by assumption, the expansion coefficients, $\alpha_{ij}$, are uniform, and the temperature change, $\Delta T$, varies linearly with the Cartesian coordinates, it follows that all these terms are zero. Hence compatibility is respected. So all the required equations are obeyed, and hence the stresses are indeed zero.

Note that the proof applies even for anisotropic materials, provided that the structure is homogeneous.

**Qu.:    If the temperature was linear in polar coordinates, would the thermal stresses be zero?**

No, not in general.

This is because when the compatibility equations are expressed in polar coordinates, there are terms involving the first derivatives as well as the second derivatives – so the above proof breaks down.

To express this in physical terms consider a cylinder. If the temperature varies linear with the radial coordinate, $r$, this will clearly cause hoop stresses because the inner and outer surfaces will want to expand by differing amounts.

Qu.:    In summary, when do we expect non-zero thermal stresses?

Thermal stresses will generally be non-zero when,

[1]  The body is unrestrained and homogeneous but subject to a temperature distribution which deviates from being linear wrt the Cartesian coordinates, or,

[2]  The body is unrestrained but has an inhomogeneous coefficient of expansion and is subject to an arbitrary temperature change (possibly uniform), or,

[3]  A homogeneous body is restrained in some way and is subject to an arbitrary temperature change (possibly uniform).

Example: Flat Plate Restrained From Bending (but free to expand) with Uniform Temperature Gradient Through-Thickness, Temperature Difference $\Delta T$ Between Surfaces

The temperatures on the surfaces are $\pm \Delta T / 2$ with respect to the middle. (Overall expansion is not restrained so we're not concerned about the mean temperature change). So the thermal strains on the surfaces are $\pm \alpha \Delta T / 2$ in all three directions (assuming an isotropic and homogeneous material). The plate is fully restrained from bending, so there is no net bending strain. Hence $\varepsilon^{el} = -\varepsilon^{th} = \mp \alpha \Delta T / 2$, where the terms relate to bending strains.

The in-plane directions, x and y, are equivalent, so surface stresses $\sigma_x = \sigma_y = \sigma$. The normal stress at the free surface is zero, $\sigma_z = 0$. Hooke's Law for the in-plane strains is thus,

$$E\varepsilon^{el} = -E\varepsilon^{th} = \sigma_x - \nu\sigma_y = (1-\nu)\sigma = \mp E\alpha\Delta T / 2$$

That is, the in-plane surfaces stresses are,

$$\sigma = \mp \frac{E\alpha\Delta T}{2(1-\nu)} \tag{15}$$

Note that the stress is tensile on the side where the temperature is lowest ($-\Delta T/2$ wrt the middle). This is an equi-biaxial stress in all in-plane directions of the plate.

Example: Thermal Shock of a Cylindrical Vessel / Pipe

A rapid change in temperature of the fluid which is in contact with the wall of the pipe causes a very thin layer of the surface of the pipe wall to cool by $-\Delta T$. It is assumed that, transiently, the bulk of the wall thickness remains unchanged in temperature. Consequently the bulk of the wall effectively fully restrains the thin layer whose temperature has dropped. The same analysis as above then gives,

$$\sigma = -\frac{E\alpha\Delta T}{(1-\nu)} \tag{16}$$

The minus sign indicates that cooling causes tension. Note that the stress in this case is double that derived above for a uniform temperature gradient. This stress is also equi-biaxial (i.e. both hoop and axial).

To be honest, not really, no.

Nevertheless, analyses are usually based on an isotropic (i.e., scalar) $\alpha$. This is OK for parent material in most cases, if grain sizes are small and their orientation is random. Then isotropic behaviour will prevail, on average, on structural scales. (What might happen at grain boundaries is another, rather interesting, question).

The most obviously anisotropic situation is for weldments. Grains are not randomly oriented in weld metal. The cooling process, proceeding from the fusion boundary inwards towards the weld centre, leads to a preferred spatial direction. For example, the weld metal grains at the fusion boundary are often columnar and oriented perpendicular to the fusion face. Weld metal can have significantly different expansion coefficients parallel to, and transverse to, these columnar grains. Most often – if weld expansion is measured at all – it will be measured on a bulk weldment specimen in directions longitudinal and transverse to the welding direction. This is not the same thing as the anisotropic expansion of the individual weld grains, being both a bulk measurement and not aligned with the individual grain axes. The HAZ can also be affected for similar reasons, though probably to a lesser extent. (Actually Young's modulus can be quite different in weld metal compared with parent, which is a further reason why thermal stresses in weld material may differ from those calculated for parent).

Since weldments are the feature of principal concern in most assessments, this issue might be significant. However, if the thermal stresses arise due to large scale temperature gradients, the weldment anisotropy might cause only a minor blip in the stress field. On the other hand, if the thermal stresses arise due to temperature gradients over a spatial scale similar to the size of the weld, then the weld anisotropy may result in significant deviations from the stresses derived assuming isotropy.

Qu.:    When is the thermal expansion coefficient inhomogeneous?

To some degree, even if anisotropy is ignored, the mean value of $\alpha$ will vary between weld and parent material, so a degree of inhomogeneity is common, though usually ignored.

The really important cases of inhomogeneous $\alpha$ are transition joints. That is, welds joining a ferritic material to an austenitic material. The former typically have an $\alpha$ in the range 12 – 14 x $10^{-6}$ /$^{o}$C, whereas the latter is more like 17 – 18 x $10^{-6}$ /$^{o}$C. This "alpha mismatch" over the fusion face creates large, but very localised, thermal stresses. R5 advises on the magnitude of alpha mismatch stress to assume (variously $E\Delta\alpha\Delta T$ or $1.5E\Delta\alpha\Delta T$, depending upon the context).

# Heat Transfer

**Qu.:** What process governs the temperature distribution in a solid?

The temperature distribution in a solid is controlled by the conduction of heat (since convection and radiation are not relevant within a solid).

**Qu.:** What is the definition of "thermal conductivity"

Heat flows from hot to cold: the steeper the temperature gradient, the faster the flux of heat. The thermal conductivity is the factor of proportionality between the two,

$$\overline{Q} = -K\overline{\nabla}T \qquad (3D) \tag{17a}$$

$$Q = -K\frac{dT}{dx} \qquad (1D) \tag{17b}$$

where $Q$ is the heat flux (energy through unit area per second), and $K$ is the thermal conductivity, and $T$ is the temperature at position $\bar{r}$ (or $x$).

**Qu.:** What is the definition of "specific heat"

The specific heat, $C_p$, is the energy required to raise the temperature of one kg of material by 1°C. If the material density is $\rho$ (in units kg/m³) the energy required to raise the temperature of 1m³ of material by 1°C is thus $C_p\rho$.

**Qu.:** What is the heat balance equation?

Assume that a certain region of a body contains no heat sources or sinks. If the temperature is increasing at a rate $\dfrac{\partial T}{\partial t}$, then the energy per unit volume of material at that point is increasing at the rate $C_p\rho\dfrac{\partial T}{\partial t}$. But the heat flowing out of this volume per second equals the divergence of the heat flux, i.e., $\overline{\nabla}\cdot\overline{Q}$, or $\dfrac{dQ}{dx}$ in 1D. In the absence of sources and sinks, the conservation of energy thus requires that a net outward flux of energy must be balanced by a reduction in the heat energy within the region, and hence a reduction of its temperature. Thus, the heat balance equation is,

$$\overline{\nabla}\cdot\overline{Q} = -C_p\rho\frac{\partial T}{\partial t} \qquad (3D) \tag{18a}$$

$$\frac{dQ}{dx} = -C_p\rho\frac{\partial T}{\partial t} \qquad (1D) \tag{18b}$$

**Qu.:** What equation governs the temperature distribution in a solid?

The heat conduction equation follows by substituting (17a) or (17b) into (18a) or (18b), giving,

$$\kappa\nabla^2 T = \frac{\partial T}{\partial t} \qquad (3D) \tag{19a}$$

$$\kappa\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad (1D) \tag{19b}$$

where the thermal diffusivity is $\kappa = \dfrac{K}{C_p \rho}$ .

**Qu.: Boundary Conditions for the Heat Conduction Equation**

The following types of boundary condition are all important,

[1] Specified, constant, boundary temperature;

[2] Specified, but varying, boundary temperature as a given function of time;

[3] Specified heat flux into/out of a surface, as a given function of time;

[4] Adiabatic boundary (i.e., insulated): This is when the heat flux at the surface is specified to be zero at all times. As a consequence of Equ.(17), this is equivalent to requiring the gradient of the temperature normal to the surface to be zero.

[*In maths speak, [3] and [4] are Neumann boundary conditions, whereas [1] and [2] are Dirichlet boundary conditions*].

[5] Convective boundary condition: In this case we do not know in advance either the temperature or the heat flux at the boundary. Instead the temperature of a fluid in contact with the boundary is specified. The heat flux into the surface of the solid is then specified via a heat transfer coefficient, $h$, as $Q = h(T_f - T_s)$, where $T_f$ is the fluid temperature, and $T_s$ is the solid temperature at its surface. Discussion of how to estimate the heat transfer coefficient, $h$, would take us firmly into the thermal analysis SQEP area. Seek advice!

[6] Radiative boundary condition: A surface will radiate heat into its surroundings in accord with Stefan's fourth-power law. Equally, the surface will absorb radiative heat from its surroundings. Thus, the net heat flux into the surface is $Q = \varepsilon\sigma\left(T_{env}^4 - T_s^4\right)$ where $\sigma$ is Stefan's constant and $\varepsilon$ is an emissivity/absorptivity factor. $T_{env}$ is the temperature of the surroundings. The temperatures in this expression *must* be absolute. In truth, a radiative boundary condition is more complicated than this because the radiation may originate from anywhere within line of sight of the surface – and this will generally involve regions of widely varying temperature. The resulting integrated effect is best left to experts.

**Qu.: Units?**

$$K = Jm^{-1}s^{-1\,o}C^{-1}; \qquad C_p = Jkg^{-1\,o}C^{-1}; \qquad \kappa = m^2 s^{-1} \qquad h = Jm^{-2}s^{-1\,o}C^{-1}$$

**Qu.: What is meant by "penetration depth"?**

Exact analytic solutions to the heat conduction equation tend to be awkward, generally involving the sum of an infinite series. But a good intuitive feel for the conduction behaviour during a thermal transient can be gained using the concept of 'penetration depth'. This is how far heat has penetrated from the surface in a given time. Be warned, this is an approximate concept only.

Since the only material property which enters the heat conduction equation, (19), is the thermal diffusivity, $\kappa$, and since its dimensions are $\kappa = m^2 s^{-1}$, dimensional analysis implies that the penetration depth must be roughly of order $d \sim \sqrt{\kappa t}$ at time $t$. A more careful study of a range of accurate solutions shows that a better approximation is,

$$d \approx \sqrt{3\kappa t} \tag{20}$$

If we consider a slab of material of a given thickness, $w$, the transit time, $\tau$, is the time taken for heat to cross from one surface to the other. In other words, it is the time when the penetration depth equals the thickness. Hence, (2) gives,

$$\tau = \frac{w^2}{3\kappa} \tag{21}$$

Note that 'transit time' is also just an approximate concept.

## Qu.:    When are transient thermal stresses significant?

Consider a vessel or pipe with an initially uniform temperature but which contains a flowing fluid whose temperature undergoes a change. This will in general produce a temperature variation through the wall of the vessel, and hence a thermal stress. But if the fluid temperature changes temperature sufficiently slowly the conduction of heat through the wall of the vessel may be able to maintain the vessel's material temperature close to uniform. In this case the thermal stresses will be small. This situation is more likely to occur for materials with high thermal conductivity because the temperature gradient depends upon the competition between the rate of heat flowing into the surface from the fluid and the rate of heat conduction away from the surface. Consequently the thermal stresses will be small if $K$ is large or $h$ is small. A dimensionless quantity which gives a feel for whether thermal transient stresses are likely to be large or small is therefore the Biot number, defined by,

$$Bi = \frac{hL}{K} \tag{22}$$

where $L$ is a characteristic dimension, often taken as the ratio of the volume of the body to the surface area in contact with the fluid. For vessels it can simply be taken to be the wall thickness.

If $Bi \ll 1$ then thermal transient stresses are expected to be small. Conversely if $Bi \gg 1$ then thermal transient stresses *might* be large (though, even then, not necessarily - as we see below). This can be seen from the following diagram, representing the temperatures during an up-shock,
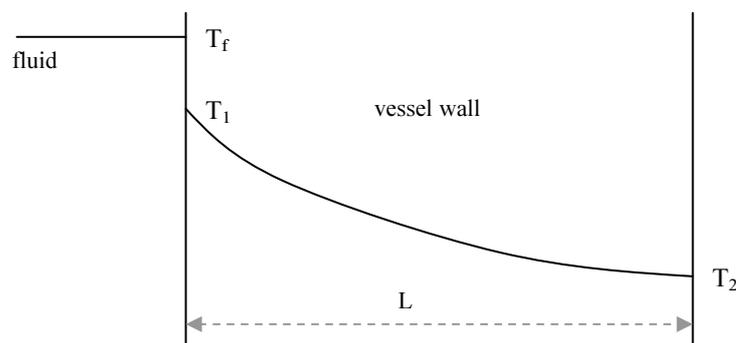


The rate of flow of heat from the fluid into the surface per unit area is $h(T_f - T_1)$, whereas the rate of flow of heat by conduction across the wall thickness is $K(T_1 - T_2)/L$. Equating these two heat flow rates gives,

$$Bi = \frac{hL}{K} = \frac{T_1 - T_2}{T_f - T_1} \tag{23}$$

From (23) we see that the temperature difference across the wall, $T_1 - T_2$, will be small if $Bi \ll 1$. Conversely, the temperature difference across the wall, $T_1 - T_2$, will be many times larger than the fluid/metal surface temperature difference, $T_f - T_1$, if $Bi \gg 1$. The latter condition does not necessarily imply large thermal stresses, however, because the rate of change of fluid temperature might be sufficiently slow that $T_f - T_1$ remains tiny at all times. We conclude the following,

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- If $Bi \ll 1$ the thermal transient stresses will be small.

- If $dT_f / dt$ is sufficiently slow then the thermal transient stresses will be small.

- If $Bi \gg 1$ and $dT_f / dt$ is sufficiently fast then the thermal transient stresses will be large.

- Whether the fluid rate-of-change-of-temperature is slow or fast may be judged from how much its temperature changes in one transit time. Thus if $\dfrac{dT_f}{dt} \cdot \dfrac{L^2}{3\kappa}$ is just a few degrees or less then the rate is slow and thermal stresses are likely to be small.

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### Example Approximate Solution for Conduction Only

Consider a slab of material of thickness $w$ which is originally isothermal (at datum zero temperature). Suppose that one surface, at $x = 0$, is raised suddenly to a temperature $T_0$ and held fixed at this temperature. The other surface (at $x = w$) is insulated (adiabatic). The gradient of the temperature distribution must therefore be zero at $x = w$. Hence we assume for simplicity that the temperature distribution is roughly parabolic, of the form,

$$T(x,t) = MAX\left[\left(1 - \frac{x}{w}\right)^2 (T_0 + T_1) - T_1, 0\right] \tag{22}$$

where the MAX part ensures that if the function on the RHS is negative then it is replaced by 0. Equ.(22) obeys the boundary conditions on both $x = 0$ and $x = w$. (But it is an approximation only, because it does not exactly obey the heat conduction equation).

The term $T_1$ is a function of time, and has yet to be determined. We find it by requiring that (22) is consistent with the penetration depth given by (20). Hence, we require,

$$\left(1 - \frac{\sqrt{3\kappa t}}{w}\right)^2 (T_0 + T_1) - T_1 = 0 \tag{23}$$

giving, for $0 \le t \le \tau$: $\quad \dfrac{T_0}{T_1(t)} = \left\{\left(1 - \dfrac{\sqrt{3\kappa \cdot t}}{w}\right)^{-2} - 1\right\} = \left\{\left(1 - \sqrt{\dfrac{t}{\tau}}\right)^{-2} - 1\right\} \tag{24}$

Using (24) within (22) gives our approximate solution.

For times later than the transit time, $t > \dfrac{w^2}{3\kappa}$, the temperature of the insulated face of the slab starts to increase and (24) is not appropriate. However, we can appeal to another physical approximation. Retaining the form (22) we can evaluate the average temperature, and hence the rate of increase of thermal energy in the body. This can be equated to the rate at which energy is flowing into the body, which is given by (17b) using the temperature gradient at x=0 from (22). This gives,

For $t > \tau$: $$\frac{T_1(t)}{T_0} = \exp\left(1 - \frac{t}{\tau}\right) - 1 \tag{25}$$

(noting that $T_1 < 0$ for $t > \tau$, and the temperature of the insulate face is $-T_1$).

This example illustrates how rough estimates of transient temperature distributions can be found using physically motivated approximations, and without solving the strict conduction equation. I don't vouch for the accuracy of (22,24,25), but these solutions do have roughly the right qualitative form – as shown by the Figure below.

A prediction from this approximate solution is that the temperature across the whole slab becomes a uniform $T_0$, to a tolerance of 2%, after 5 times the transit time, i.e., after the time $t_{uniform} = \dfrac{5w^2}{3\kappa}$. I'd be interested to know if this is accurate.

**Figure**

Approximate solution, Equs.(22,24,25), for the temperature distribution through a slab of material whose surface undergoes a step change in temperature at time zero: Temperature normalised by the surface temperature change ($T/T_0$) versus normalised position ($x/w$). Each curve corresponds to a different time, as shown in the legend normalised by the transit time.
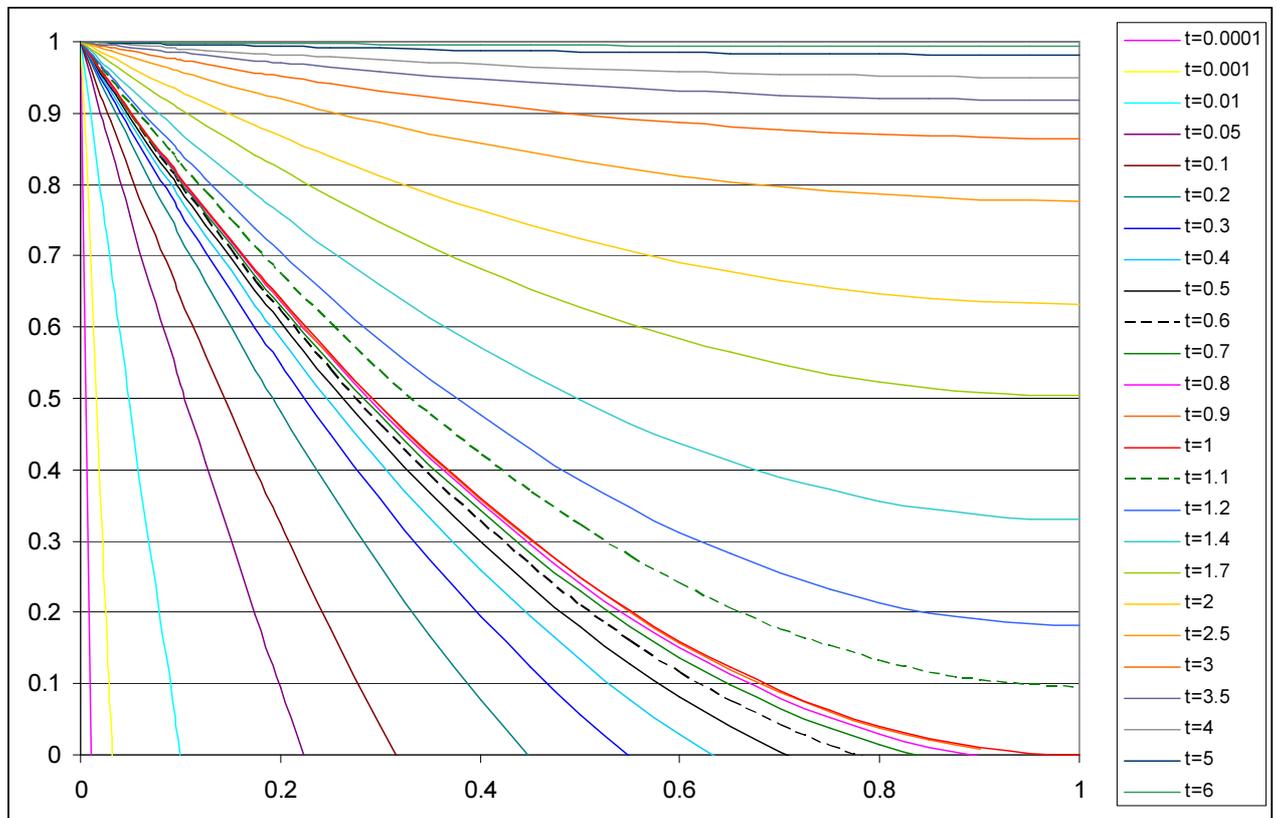


Qu.:    What is a "steady state" temperature distribution?

The steady state temperature distribution is simply what the temperature distribution tends to after a long time. If the boundary conditions continue to change indefinitely, for example if we apply a sinusoidally varying surface temperature, then there will be no steady distribution.

The simplest example of a steady state distribution occurs when all the boundary conditions are prescribed, constant temperatures. A qualitative picture of what the steady state temperature distribution will look like can be constructed as follows,

- Draw lines of heat flux, flowing from the hot boundaries to the cool boundaries;

- The isotherms (lines of constant temperature) are then drawn by following the rule that they are everywhere perpendicular to the lines of heat flux.

For a 1D problem, the steady state is a linear distribution (i.e., a uniform temperature gradient).