Tutorial Session 10 T72S01
Relates to T72S01 Knowledge & Skills 1.15, 1.18
Desirable but not essential for SQEP.
The upper bound theorem of plastic collapse: proof and example applications.
Last Update: 22/4/14

Qu.: What is the Upper Bound Theorem of plasticity?

The Upper Bound Theorem: An estimate, P, of the collapse load of a structure is made by,

(a) Postulating a 'mechanism' (a kinematically consistent distribution of displacement increments, du_i, and plastic strain increments, $d\varepsilon_K^p$);

(b) Evaluating the work done within the structure by the postulated plastic strain increments together with the corresponding stress on the yield surface;

(c) Equating the work done by the load P to the internal work done by the strains within the structure.

Such an estimate is an upper bound to the true collapse loads, P.

- Kinematic" just means "geometrical". So a kinematically consistent mechanism is one in which the strain rates and displacement rates are related as they should be, that is, $\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2$.
- The phrase "the corresponding stress on the yield surface" means the deviatoric stress at which the postulated plastic strain increment is normal to the yield surface. At all points where plastic strain rates are assumed, the equivalent stress must equal yield.
- Like the lower bound theorem, the upper bound theorem applies only to elastic-perfectly plastic material (i.e., material with no work hardening). In practice this is ignored and it is applied anyway – but exercising caution in regard to the effective 'yield' stress.

Qu.: Can the Theorem be stated more simply?

I'm not sure if it can be stated more simply whilst also being rigorous, but perhaps a more readily appreciated statement, though less precise, is,

The Upper Bound Theorem: Any mechanism which permits unbounded displacements whilst being consistent with the yield surface and the conservation of energy must already be beyond the collapse load.

Qu.: What use is the upper bound theorem?

Since it finds an *upper* bound to collapse, it may not provide a *safe* reference stress to use in R5, R6, etc. However, it can do so if it is possible to find a "minimum upper bound" assuming some mode of deformation. This should then be accurate as long as the deformation mode is reasonable – and hence could be used in assessments.

Historically, the upper bound approach to estimating a collapse load has been used when it is conservative to err on the high side – for example, when designing a machine to roll or extrude ingots in a mill. In this case you want to ensure that the mill has the capacity to do its job, and so you will want an upper bound of the ingot's 'collapse' load (i.e., the load to deform it into the new billet).

Qu.: What role does equilibrium play in the upper bound theorem?

None. Equilibrium is not necessarily respected by the postulated mechanism.

Qu.: What role do displacements and strains play in the upper bound theorem?

Strictly we mean displacement increments and strain increments (or rates). These are crucial since they define the kinematic mechanism. The displacement rates and strain rates must be compatible, so as to provide a consistent kinematic mechanism. Otherwise you can postulate them arbitrarily.

Qu.: What yield criterion is required for the upper bound theorem to be true?

Different equivalent stresses can be chosen according to which theory of plasticity one favours. More generally the condition that "the equivalent stress does not exceed yield anywhere" can be replaced by "the yield condition is not violated anywhere". In other words, it may be Tresca, or Mises, or anything else. The theorem is not restricted to what yield condition may pertain for the material. Like the lower bound theorem, the upper bound theorem relies only upon the convexity of the yield surface and upon the normality of the plastic flow, not upon any specific details beyond that.

Qu.: What types of applied load does the upper bound theorem relate to?

The load P may symbolically represent any combination of point loads, pressures, tractions, body forces etc, as long as they are all *load controlled* loads.

Qu.: What do we mean by "an upper bound of P" if P comprises several loads?

Equating the external and internal work done gives one equation which permits an upper bound to be found for one load variable. The values for the other loads must be assumed. This means that just one of the individual loads is found and justified to be an upper bound, given the assumed values of the other loads.

Another way of viewing this is that a surface in load-space is provided by the work equation, and the true collapse surface lies within it.

Qu.: What if the guessed mechanism is a really bad guess?

If the guess is poor, the loads P may be a uselessly high upper bound.

Qu.: What corollaries follow from the upper bound theorem?

The same corollaries follow from the upper bound theorem as from the lower bound theorem, namely,

- Removing material* from a body cannot increase its plastic collapse load.
- Adding material* to a body cannot reduce its plastic collapse load.

**assuming the added weight is negligible.*

This can be seen as follows: suppose a body is collapsing. Now consider the body with some region removed. The actual collapse mechanism provides a postulated mechanism for the body which remains after material has been removed. But since the internal rate of work associated with the removed material does not now contribute, the load obtained from the work equation is reduced (or, at best, remains unchanged if the material removed took not part in the mechanism). Moreover, this is an upper bound to the collapse load of the reduced body, and hence its true collapse load must be reduced (or remain unchanged at best). **QED**.

The second corollary is just the logical negative of the first.

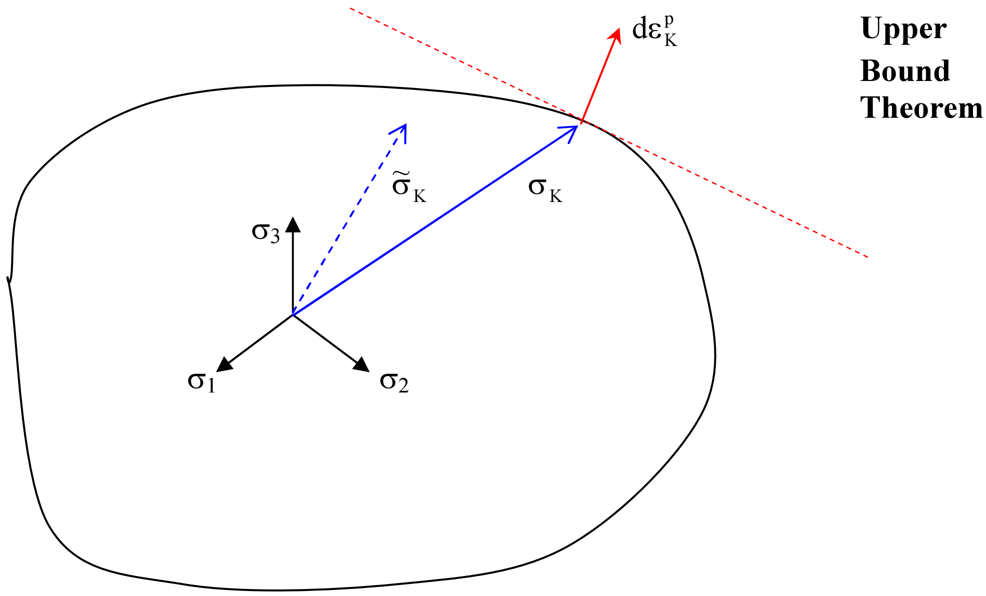
Qu.: But that's just trivial, isn't it?

No! These obvious sounding theorems are <u>not</u> true for failure by fracture, since,

- Adding a block of material containing a crack can reduce the load carrying capacity of the structure, and,
- Removing material from around a crack tip can increase the load carrying capacity of the structure.

Qu.: What is the proof of the upper bound theorem?

The situation is illustrated by a diagram very similar to that used to prove the lower bound theorem:-



The stress with the tilda is the true collapse stress;

The plastic strain increment shown is postulated, consistent with the mechanism of deformation of the body;

The stress without the tilda is that which is required to be consistent with the mechanism, i.e., lying on the yield surface where its normal is parallel to the strain rate

Suppose the true collapse loads, $\tilde{P}_i$, are a factor λ times the postulated collapse loads, P_i, i.e., $\tilde{P}_i = \lambda P_i$. We are attempting to show that $\lambda \leq 1$. In the upper bound theorem the loads, P_i, are required to satisfy the balance of energy, i.e.,

$$\sum_i P_i du_i = \int \sigma_K d\varepsilon_K^p dV \tag{1}$$

all the quantities in Equ.(1) being *postulated* rather than actual values. Applying the virtual work argument to the actual stress distribution, $\tilde{\sigma}_K$, in equilibrium with the actual collapse loads, $\tilde{P}_i$, together with the postulated mechanism gives,

$$\sum_i \tilde{P}_i du_i = \int \tilde{\sigma}_K d\varepsilon_K^p dV \tag{2}$$

But from the diagram we have that,

$$\tilde{\sigma}_K d\varepsilon_K^p \leq \sigma_K d\varepsilon_K^p \qquad (3)$$

at all points in the body. From (1) and (2) it thus follows that,

$$\sum_i \tilde{P}_i du_i = \lambda \sum_i P_i du_i \leq \sum_i P_i du_i \qquad (4)$$

i.e., $\lambda \leq 1$

Thus, our postulated loading is an upper bound to the true collapse load. **QED**.

Qu.: Why does the elastic energy play no part in this proof?

The alert will have noticed that the proof assumes that the increments of elastic strain, $d\varepsilon_{ij}^{el}$, are zero at collapse, and hence that the elastic strain energy increment is zero as collapse is approached. That is, the elastic regions are assumed to behave rigidly at collapse. This is where the assumption of perfect plasticity (no hardening) is important. In perfect plasticity, the stresses will be unchanging as plastic flow proceeds when close to collapse. Since the stresses are unchanging, the elastic strains are also unchanging, as assumed. But for a material which is still hardening even as collapse is approached, the increasing elastic strain energy would screw-up the proof. Hence the assumption of perfect plasticity.

Qu.: How is the Upper Bound Theorem applied in practice?

Most often the mechanism is defined by considering the body to comprise a number of rigid (elastic) regions which can slip with respect to each other along their boundaries. These 'slip lines' represent regions of intense (in fact divergent) shear strain, but of small (in fact zero) volume. The work done is simply the area of the slip line times the shear yield stress (which gives the tangential force) times the distance slipped.

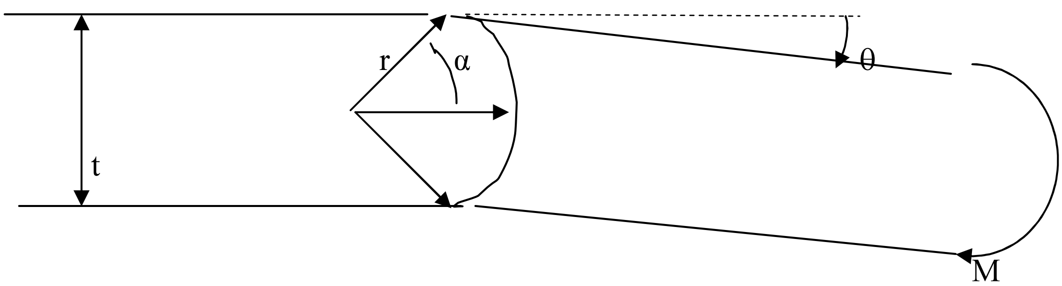
Because mechanisms of this type involve only shear, and hence only the shear yield stress, and because $\tau_y = \sigma_y / 2$ for Tresca but $\tau_y = \sigma_y / \sqrt{3}$ for Mises, these solutions always produce a Mises collapse load which is $2/\sqrt{3}$ larger than the Tresca result.

Note that the 'slip lines' used to define these mechanisms for upper bound purposes are not the same as the 'slip line fields' that will be considered in the next session.

Some typical examples applying the upper bound method follow…

(1) Rectangular Section Bar In Bending (Upper Bound)

Inspired by our knowledge of the true plastic hinge, we postulate that slip occurs along a circular arc of some unknown radius 'r',



Geometry: $2r\sin\alpha = t$, Slip distance = rθ. Slip line area = 2Brα

Hence, work done = $M_y\theta = 2Br\alpha\tau_y.r\theta = \dfrac{Bt^2\theta\tau_y}{2}\cdot\dfrac{\alpha}{\sin^2\alpha}$ (5)

We are free to choose the angle α (or equivalently, the radius r) as we wish. Since Equ.(5) gives an upper bound for the collapse moment, we wish to minimise it to get the most accurate result (i.e., closest to reality). The minimum of the function $\dfrac{\alpha}{\sin^2\alpha}$ is easily shown to be 1.38 at an angle of $\alpha = 67^o$. Hence our "least upper bound" collapse solution (for this particular mechanism) is,

$$M_y = \frac{1.38Bt^2\tau_y}{2} = \frac{1.38Bt^2\sigma_y}{4}\text{(Tresca)} = \frac{1.38Bt^2\sigma_y}{2\sqrt{3}}\text{(Mises)} \qquad (6)$$

Thus, this upper bound result is between 38% and 59% larger than the lower bound result, depending upon the yield theory used (i.e. a factor of 2.07 to 2.39 times the moment to first yield, compared with 1.5 for the lower bound result). ***BUT it's an upper bound, so it would be non-conservative to use in an assessment***. There are better (i.e., smaller) upper bound solutions.

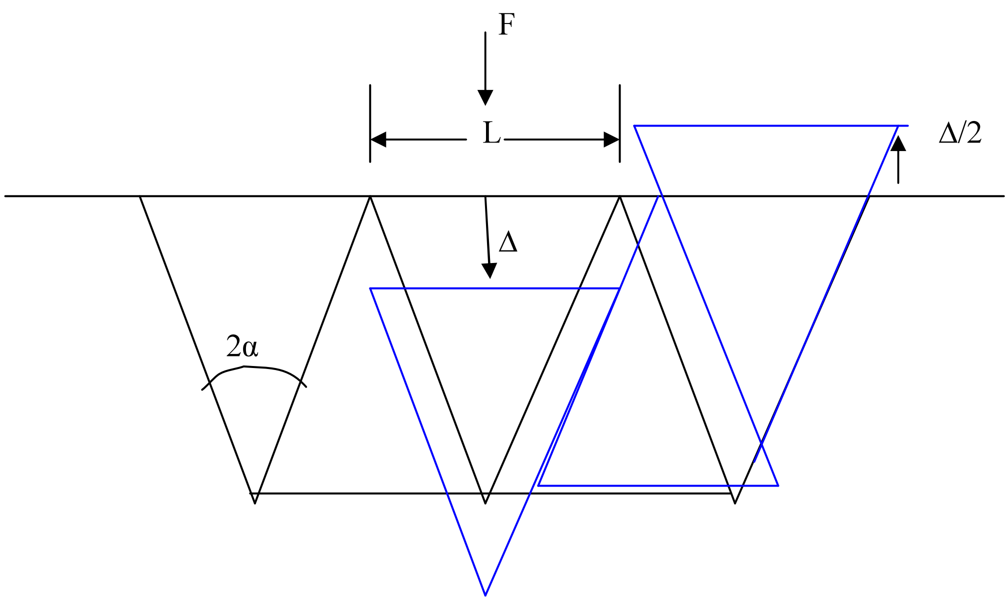
The upper bound result 'predicts' a reasonable shape for the plastic hinge.

NB: It makes no difference if we include the other side of the plastic hinge. Each side ('arc') of the plastic hinge is merely subject to half the angular displacement, and hence the result is the same.

---

(2) Indentation of a Semi-Infinite Slab (Upper Bound)

A hard flat indenter of width L is pressed into the surface of a semi-infinite flat slab of material of shear yield strength τ_y. The indenter is assumed to be sufficiently hard that it is undeformed. A mechanism is postulated consisting of five sliding isosceles triangles, each of whose apex angles is 2α, i.e.,



The blue lines show the displaced triangles – on one side only – due to a downwards displacement of the indenter by Δ. *(Apologies for the distortions – my skill with WordArt is poor).*

Note: Do not be perturbed by the fact that the mechanism looks impossible due to the corners of some triangles penetrating into solid material. The volumes of these regions are of second order in the small displacement, Δ. It can be shown that if the slipping zones are given a finite width the regions of overlap disappear.

Elementary geometry gives the horizontal sliding of four of the five triangles to be $\Delta \tan\alpha$. Similarly it is easily seen that the upward displacement of the outer triangles is Δ/2. (Consequently the volume of the indent equals the volume of the extruded material, as it should). The slip distance along the first interface is $\Delta / \cos\alpha$. The slip distance along the other 'long' edges is half this. Also, the long side of the triangles is $L / 2\sin\alpha$.

For a width B into the plane of the paper, the work done by the slipping zones (i.e. the sum of the products of the areas of the slipping lines, $\times \tau_y$, times their slip distance) is,

$$\begin{aligned} \text{Work Done} &= 2B\tau_y \left\{ \frac{\Delta}{\cos\alpha} \cdot \frac{L}{2\sin\alpha} + 2 \cdot \frac{\Delta}{2\cos\alpha} \cdot \frac{L}{2\sin\alpha} + \Delta \tan\alpha \cdot L \right\} \\ &= 2B\Delta L\tau_y \left\{ \frac{1}{\sin\alpha\cos\alpha} + \tan\alpha \right\} \end{aligned} \quad (7)$$

We must equate this to the work done by the indenter, i.e. FΔ. Hence, the upper bound yield pressure is,

$$\frac{F}{BL} = 2\tau_y \left\{ \frac{1}{\sin\alpha\cos\alpha} + \tan\alpha \right\} \quad (8)$$

Choosing equilateral triangles ($\alpha = 30^o$) this becomes,

$$\frac{F}{BL} = \frac{10\tau_y}{\sqrt{3}} = 5.77\tau_y = 2.89\sigma_y (\textit{Tresca}) \text{ or } 3.33\sigma_y \text{ (Mises)} \quad (9)$$

The best solution for our assumed mechanism is given by finding the *minimum* of the above function of angle, {…} in Equ.(8), which is 2.828 for $\alpha = 35.2^o$. This is actually very little different from the equilateral triangle case,

$$\frac{F}{BL} = 5.66\tau_y = 2.83\sigma_y (\textit{Tresca}) \text{ or } 3.27\sigma_y \text{ (Mises)} \quad (10)$$

Assuming that this is not a wildly poor estimate (it isn't), this implies that the "yield" stress of a material could be estimated from a hardness test by multiplying the average indenter pressure by 0.3. Of course, this would be a strain hardened "yield" stress, corresponding to some strain representative of the indentation depth. I don't suggest that it's an accurate method – but it may be of use if you only have a tiny sample of material.

(3) Collapse Pressure for a Pipe Pressurised Internally and Externally (Tresca)

NB: Although external pressure is considered here, only the plastic collapse mechanism is addressed. The buckling mechanism is not considered. This would be necessary in a complete assessment.

Here we depart from the normal practice of defining the mechanism via rigid regions and slip lines. Instead we postulate a continuum of displacements and strains.

By axisymmetry the strains can be written $\varepsilon_r = \frac{du_r}{dr}$ and $\varepsilon_\theta = \frac{u_r}{r}$, and $\varepsilon_{r\theta} = 0$. The upper bound theorem requires any compatible system of displacement rates and strain rates to be postulated, together with stresses which are on the yield surface where-ever the strain rates are non-zero. So, we assume that the axial strain rate is zero. This means that $\dot{\varepsilon}_r + \dot{\varepsilon}_\theta = 0$ by plastic incompressibility, and hence $\frac{d\dot{u}_r}{dr} = -\frac{\dot{u}_r}{r}$. Integrating gives $\dot{u}_r = \frac{A}{r}$ and $\dot{\varepsilon}_\theta = -\dot{\varepsilon}_r = \frac{A}{r^2}$ for some constant, A.

Since the strain rates are non-zero everywhere, the equivalent stress must equal yield everywhere, i.e., $\sigma_\theta - \sigma_r = \sigma_y$ assuming the Tresca criterion. (This assumes that the axial stress must be intermediate between the hoop and radial stresses. This follows from the fact that the strain rates obey $\dot{\varepsilon}_\theta > \dot{\varepsilon}_A > \dot{\varepsilon}_r$, because $\dot{\varepsilon}_\theta > 0$, $\dot{\varepsilon}_A = 0$, $\dot{\varepsilon}_r < 0$, and the normality rule).

The rate of work done per unit volume is $\xi = \sigma_\theta \dot{\varepsilon}_\theta + \sigma_r \dot{\varepsilon}_r = (\sigma_\theta - \sigma_r)\dot{\varepsilon}_\theta = \sigma_y \dot{\varepsilon}_\theta$. There is no contribution from the axial stress because we have chosen the axial strain rate to be zero. Hence, $\xi = \frac{A\sigma_y}{r^2}$. The total rate of work is found by integrating ξ times the element of volume, which is $2\pi r dr$ per unit axial length. Hence, the work rate is,

$$\int_a^b \frac{A\sigma_y}{r^2} 2\pi r dr = 2\pi A \sigma_y \log\left(\frac{b}{a}\right) \tag{11}$$

But the rate at which the internal pressure does work, per unit axial length, is $2\pi a P_i$ times the radial displacement rate at the bore, i.e., $\dot{u}_r(r = a) = \frac{A}{a}$, which gives $2\pi A P_i$ since the 'a' terms cancel[1]. The external pressure does negative work if the displacement is outwards, and hence overall the rate at which the applied loads do work is $2\pi A(P_i - P_e)$. Equating this with the above expression gives,

$$P_i - P_e = \sigma_y \log\left(\frac{b}{a}\right) \tag{12}$$

This establishes that this is an upper bound to the collapse load. Since we already know from session 9 that this is also a lower bound solution, it must be the exact solution.

It can be shown that the upper bound (and indeed, the exact) Mises solution is the usual factor $2/\sqrt{3}$ times (12) – although this is not obvious in this case.

---

[1] Note that the axial pressure load does no work, since, by assumption, there is no axial strain rate.

Qu.: Is P_y the collapse load or the general yield load?

That depends upon what material property, σ_y, has been used in its definition. If this is the 0.2% proof stress, then P_y is really the yield load (and, more precisely, the yield load at the 0.2% level). But if some hardened stress is used, e.g. the flow stress, then P_y might loosely be called the plastic collapse load. In truth it is probably most accurate to use the UTS here, to get the best estimate of the collapse load. But, for conservatism, codes like R6 recommend a flow stress defined as the average of the 0.2% proof stress and the UTS.