**SQEP Tutorial Session 1: T72S01**
**Relates to Knowledge & Skills 1.4**
Last Update: 10/9/13

*Bending stresses: Formulae relating stress to bending moment; Formulae relating stress to radii of curvature; Section modulus and second moment of area; Formulae for arbitrary sections; Cross-product moment of area; Neutral axis for asymmetric sections*

Notation:-
$M$ = bending moment (units Nm);
$\tilde{M}$ = bending moment per unit length (units Nm/m = N)

Qu.: What is bending stress?

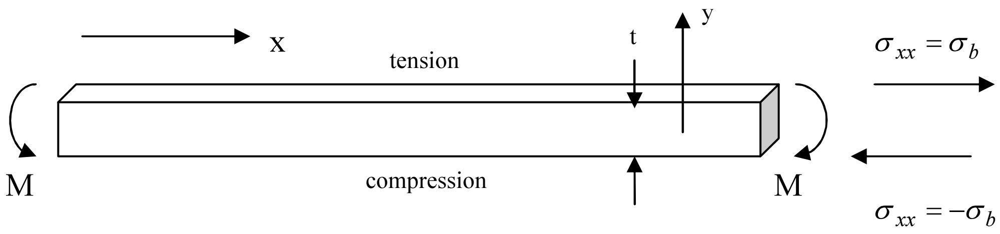
Bending stress is the part of a stress distribution which varies linearly across a section and which integrates to zero net force. Bending stress is what results from an applied bending moment.

Qu.: What is the "outer fibre stress"?

Because a bending stress varies linearly across a section, it has its greatest magnitude on one of the surfaces. For symmetrical sections, the furthest points from the middle of the section, on either side, are referred to as the 'outer fibres'. For arbitrary sections and a bending moment applied about an arbitrary axis in the plane of the cross-section, there is always a "neutral axis" which is the line along which the stress is zero. The "outer fibres" are the points with the greatest perpendicular distance from the neutral axis, on either side. The maximum stress, on the outer fibre, is simply called 'the bending stress'.

Qu.: Which stress component is a bending stress?

If the beam lies along the x-axis, then the bending stress is the x-stress, $\sigma_{xx}$. Thus,




The stress at a distance $y$ from the neutral axis is $\sigma_{xx} = \frac{2y}{t}\sigma_b$, if the section is symmetrical and of depth $t$ (noting that the origin is at the centre of the section).

Qu.: What is the 'neutral axis'?

The neutral axis is the line along which the stress is zero. It always passes through the centroid of the section.

For a section with a plane of symmetry, and for a bending moment applied on an axis lying in or perpendicular to this plane of symmetry, the neutral axis is the line parallel to the axis of the bending moment and passing through the centroid.

Qu.: What is the bending moment for a bending stress over a rectangular section?

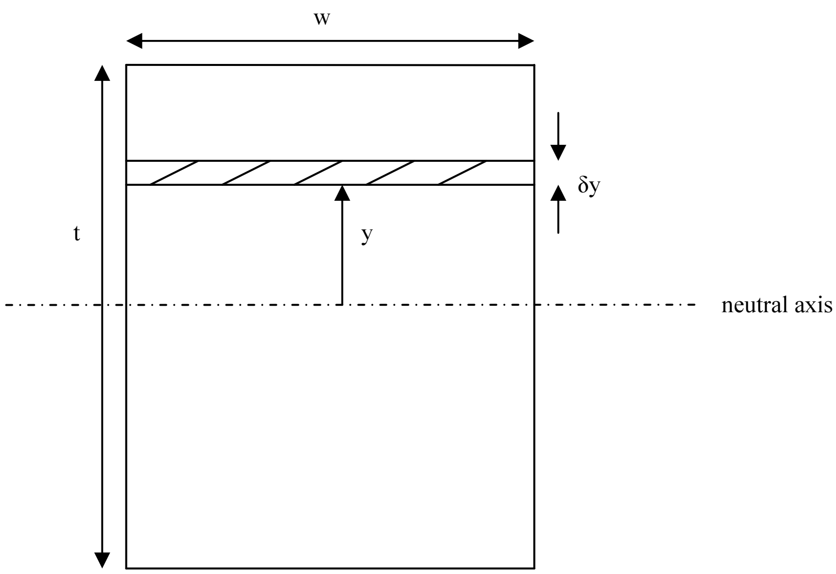
Suppose the beam depth is $t$ and its width is $w$. The relation between applied bending moment and bending stress is $\sigma_b = \frac{6M}{wt^2}$. This is one to remember.

**Proof:** The moment is found by integrating the contributions from each slice across the beam (see diagram below). The area of the shaded slice is $w\delta y$, hence the force over this area is $\sigma_{xx} w \delta y$. The bending moment about the neutral axis is thus $\delta M = y\sigma_{xx} w\delta y$. Substituting the value for the x-stress in terms of the position, $y$, gives,

$$\delta M = y\frac{2y}{t}\sigma_b w\delta y$$

Hence, $$M = \frac{2w\sigma_b}{t}\int_{-t/2}^{+t/2} y^2 \delta y = \frac{2w\sigma_b}{t}\cdot\left.\frac{y^3}{3}\right|_{-t/2}^{+t/2} = \frac{2w\sigma_b}{3t}\left(\frac{t}{2}\right)^3 \times 2 = \frac{wt^2\sigma_b}{6}$$

which gives, $$\sigma_b = \frac{6M}{wt^2} \qquad \text{QED.} \tag{1}$$




Qu.: Is this formula relevant to anything other than rectangular section beams?

Yes. Suppose a shell (of any shape) has a through-wall bending moment per unit length of $\tilde{M}$. In the above formula this is just $\tilde{M} = M / w$, so we immediately get the relationship between bending stress and the moment resultant, $\tilde{M}$, for any shell to be $\sigma_b = \frac{6\tilde{M}}{t^2}$. This is one to remember. (2)

Qu.: Where is the axis of bending if the section is not symmetrical?
(For the time being we are assuming there is a vertical plane of symmetry)

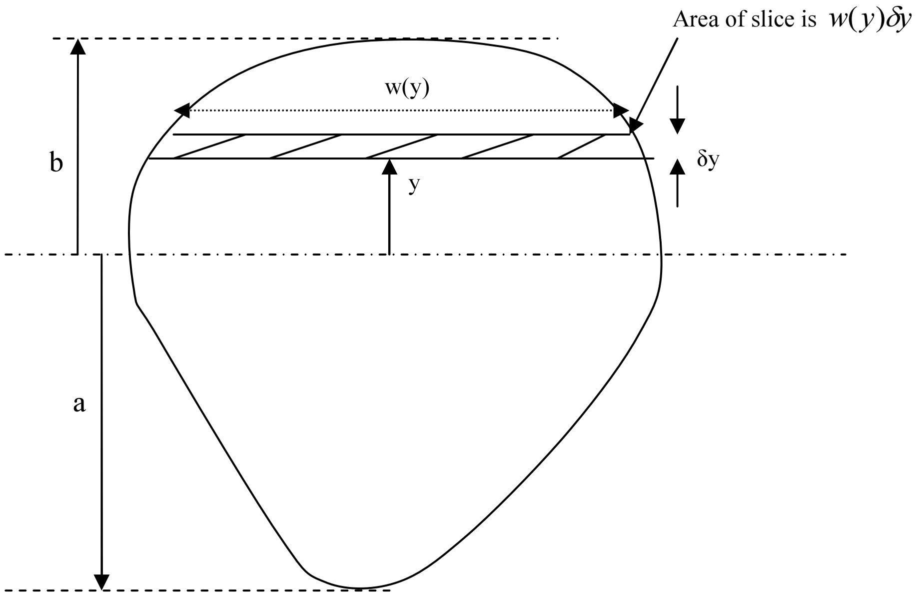
The bending stress at a distance $y$ from the neutral axis is $\sigma_{xx} = Cy$, for some constant $C$, but we do not initially know where the origin of $y$ is positioned. However, the net force must be zero. The small element of force corresponding to the thin slice shown shaded in the diagram below is, $\delta F = \sigma_{xx} w(y)\delta y = Cyw(y)\delta y$. The difference from the rectangular section is that, for an arbitrary section, the width, $w$, depends upon the height, $y$, above the neutral axis (see diagram). This is explicit in the notation which shows $w$ to be a function of $y$, i.e., $w(y)$. The requirement that the total force be zero gives,

$$F = C \int_{section} yw(y)\delta y = 0$$

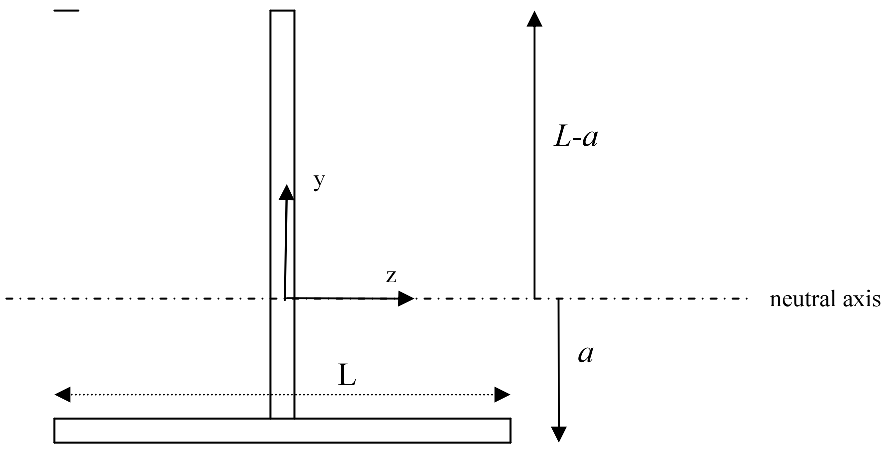
But $w(y)\delta y$ is the area of the thin slice, $\delta A$. So this can be written,

$$\int_{section} ydA = 0 \qquad (3)$$

This is the requirement which defines the position of the neutral axis (i.e., the assumed origin of the $y$ coordinate).



Qu.: Example: Where is the neutral axis of a T-section?




Both legs are of length $L$ and thickness $t$, and for simplicity we shall work in the approximation $t << L$. We wish to find the position, $a$, of the neutral axis above the bottom leg. The contribution to $\int ydA$ of the bottom leg is approximately $-aLt$,

negative because it lies below the neutral axis. The contribution of the vertical leg is $\int ydA = \int_{-a}^{L-a} ytdy = \left.\frac{ty^2}{2}\right|_{-a}^{L-a} = \frac{t}{2}\left[(L-a)^2 - a^2\right] = \frac{t}{2}\left(L^2 - 2aL\right)$. Adding the contribution of

the horizontal leg gives a total $\frac{t}{2}\left(L^2 - 4aL\right)$ which must be zero. Hence, $a = L/4$.

Qu.: Why does the neutral axis pass through the centre of gravity (centroid)?

The centroid (cog) is defined by zero net moment due to an assumed uniform weight per unit area (with gravity in the x-direction). This leads to the requirements $\int_{section} ydA = 0$ and $\int_{section} zdA = 0$. But these are the same as is required to ensure that a bending stress (i.e., a stress varying linearly across the section) produces zero net force. QED.

Qu.: What is the relation between bending moment and bending stress?
(For the time being we are assuming there is a vertical plane of symmetry)

Consider the irregular section in the above diagram, for which the small force due to the thin slice is $\delta F = \sigma_{xx} w(y)\delta y = Cyw(y)\delta y = Cy\delta A$. The moment about the neutral axis due to this force is $\delta M = Cy^2\delta A$, hence the total moment is,

$$M = C \int_{section} y^2 \delta A = CI$$

where the "second moment of area" (sometimes improperly called the 'moment of inertia') has been defined by,

$$I = \int_{section} y^2 dA \qquad (4)$$

If the outer fibre on the bottom of the beam is a distance $a$ below the neutral axis, and the outer fibre on top is a distance $b$ above the neural axis, then we can define the bending stress at either location, i.e., as $\sigma_b = Cb$, or as $-\sigma_b = -Ca$. (I don't think there is an agreed convention). Assuming the former, we can replace the constant $C$ by $C = \sigma_b / b$, and hence we get the general relation between bending stress and bending moment for an arbitrary section,

$$M = \frac{I\sigma_b}{b} \qquad \text{or,} \qquad \sigma_b = \frac{Mb}{I} \tag{5}$$

This is the most important equation to remember in this topic.

Qu.: What is the "section modulus"?

A terminology which is also used is 'section modulus', defined as

$$S = I / b \tag{6}$$

so that $\sigma_b = \dfrac{M}{S}$.

Qu.: Example: What is the section modulus for a rectangular section?

Answer: $I = \dfrac{wt^3}{12}$ and $b = \dfrac{t}{2}$ so $S = \dfrac{wt^2}{6}$ (7)

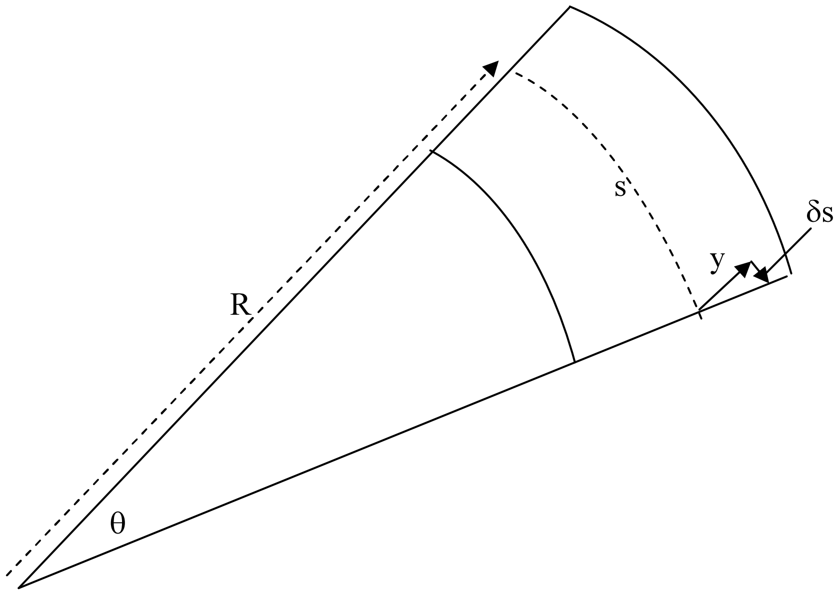
**Proof**: $I = \int_{section} y^2 dA = \int_{-t/2}^{+t/2} y^2 w dy = w \dfrac{y^3}{3}\Big|_{-t/2}^{+t/2} = \dfrac{w}{3}\left(\dfrac{t}{2}\right)^3 \times 2 = \dfrac{wt^3}{12}$ QED.

This retrieves the bending stress formula for a rectangular section since,

$$\sigma_b = \frac{Mb}{I} = M \cdot \frac{t}{2} \cdot \frac{12}{wt^3} = \frac{6M}{wt^2}.$$

Qu.: What is the constant of proportionality in $\sigma_{xx} = Cy$ in terms of strain?

The constant of proportionality is simply related to the radius of curvature of the beam. In the $x, y$ plane the deformation of a segment of the beam (exaggerated) may look like,




The bending strain at a distance $y$ from the neural axis is $\varepsilon = \dfrac{\delta s}{s}$.

But the angle defining the segment of beam pictured can be written in two ways, so that $\theta = \dfrac{s}{R} = \dfrac{\delta s}{y}$. Hence $\varepsilon = \dfrac{\delta s}{s} = \dfrac{y}{R}$ where $R$ is the radius of curvature.

But for simple uniaxial stressing, as here, the relationship between stress and strain is just $\sigma = E\varepsilon = \dfrac{Ey}{R}$ where $E$ is Young's modulus. (8)

This establishes that the proportionality constant in $\sigma = Cy$ is $C = E/R$.

The reciprocal of the radius of curvature is the curvature, so $C$ is the curvature multiplied by $E$.

Qu.: Are the formulae developed so far the full story?

No.

The simple formulae $\sigma = \frac{Ey}{R}$ and $\sigma_b = \frac{Mb}{I}$ assume that the bending deformation occurs about the same axis as the applied bending moment.

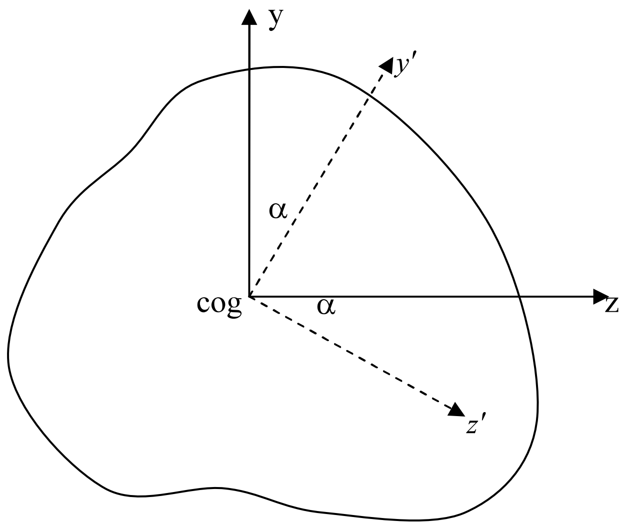
This will be true if the section has a plane of symmetry and the bending moment is applied either about this plane or perpendicular to this plane.

But, in general, a moment applied about an arbitrary axis $\hat{m}$ in the plane of the cross-section will result in bending about a different axis.

Hence, even for a section with a plane of symmetry, like the T-section illustrated above, if the axis of the applied moment, $\hat{m}$, is neither horizontal nor vertical, the axis of the bending deformation will differ from $\hat{m}$.

To develop the theory for arbitrary sections and arbitrary orientations of applied moment we need to take into account the bending which occurs perpendicular to the applied moment.

Qu.: How are moments and bending related for arbitrary sections?




The beam is oriented along the x-axis so that its cross-section is the $(y, z)$ plane.

The stress $\sigma$ will be understood to refer to the x-stress ($\sigma_{xx}$).

$M_y, M_z$ are the moments about axes $y$ and $z$ respectively.

$R_y, R_z$ are the radii of curvature in the $(x, y)$ and $(x, z)$ planes respectively.

The second moments of area about the y and z axes are,

$$I_z = \int y^2 dA \quad \text{and} \quad I_y = \int z^2 dA \tag{9}$$

where it is understood that the origin of the y and z axes is at the centroid of the cross-section.

If bending deformation were confined to the $(x, y)$ plane then,

$$\sigma = \frac{Ey}{R_y} \tag{10}$$

The moments are thus,

$$M_z = \int \sigma y \cdot dA = \frac{E}{R_y}\int y^2 dA = \frac{EI_z}{R_y} \tag{11a}$$

and,
$$M_y = \int \sigma z \cdot dA = \frac{E}{R_y}\int yz \cdot dA = \frac{EI_{yz}}{R_y} \tag{11b}$$

where we have also defined the cross-product moment of area by,

$$I_{yz} = \int yz \cdot dA \tag{12}$$

Note that despite the bending being about the z-axis, there is a non-zero moment about the y-axis in the general case, given by (11b).

If we now also allow bending deformation in the $(x, z)$ plane, superposition will clearly give the stress and moments as,

$$\sigma = \frac{Ey}{R_y} + \frac{Ez}{R_z} \tag{13}$$

$$M_z = \frac{EI_z}{R_y} + \frac{EI_{yz}}{R_z} \tag{14a}$$

and,
$$M_y = \frac{EI_{yz}}{R_y} + \frac{EI_y}{R_z} \tag{14b}$$

The moment equations can be inverted to give the radii of curvature,

$$\frac{1}{R_z} = \frac{1}{E\|I\|}\left(I_z M_y - I_{yz} M_z\right) \tag{15a}$$

$$\frac{1}{R_y} = \frac{1}{E\|I\|}\left(I_y M_z - I_{yz} M_y\right) \tag{15b}$$

where,
$$\|I\| = I_y I_z - I_{yz}^2 \tag{15c}$$

Equs.(15) show explicitly that bending deformation occurs perpendicular to the applied bending moment if the cross-product moment of area, $I_{yz}$, is non-zero, e.g., $1/R_z$ is non-zero for an applied $M_z$ when $M_y = 0$ if $I_{yz} \neq 0$. Substituting (15a,b) into (13) gives the general form of the moment-stress relationship,

$$\sigma = \frac{y\left(I_y M_z - I_{yz} M_y\right) + z\left(I_z M_y - I_{yz} M_z\right)}{\|I\|} \tag{16}$$

Note that this gives the stress at any $y, z$ coordinate lying within the cross-section. The bending stress is defined as the maximum value of (16), or possibly the maximum compressive value.

Qu.: Where is the neutral axis in general?

The neutral axis is defined as the axis along which the stress is zero. The stress varies linearly with perpendicular distance from the neutral axis. Define axes $y', z'$ rotated clockwise by an angle $\alpha$ with respect to $y, z$ (see above Figure). Hence,

$$y' = y\cos\alpha + z\sin\alpha \tag{17a}$$

$$z' = z\cos\alpha - y\sin\alpha \tag{17b}$$

If we choose $z'$ to be the neutral axis, then the stress should be proportional to $y'$ so that comparison of (17a) with (16) gives,

$$\left(I_y M_z - I_{yz} M_y\right) = D\cos\alpha \tag{18a}$$

$$\left(I_z M_y - I_{yz} M_z\right) = D\sin\alpha \tag{18b}$$

for some constant $D$. Hence,

$$\tan\alpha = \frac{\left(I_z M_y - I_{yz} M_z\right)}{\left(I_y M_z - I_{yz} M_y\right)} \tag{19}$$

and $\alpha$ is the angle between the neutral axis and the horizontal. Substituting (18a,b) into (16) the stress can be written more simply as,

$$\sigma = \frac{Dy'}{\|I\|} \tag{20a}$$

where, $$D = \sqrt{\left(I_y M_z - I_{yz} M_y\right)^2 + \left(I_z M_y - I_{yz} M_z\right)^2} \tag{20b}$$

This establishes that the stress does indeed vary in proportion to the perpendicular distance from the neutral axis ( $y'$ ), as claimed. Equ.(20a) is the generalisation of the simple formula $\sigma_b = \frac{My}{I}$. The "outer fibre" is the point which has the maximum (or algebraic minimum) $y'$, and hence the maximum (or maximum compressive) stress.

Qu.: What about sections with a plane of symmetry?

It can immediately be seen that $I_{yz} = \int yz \cdot dA$ will be zero if either the y or the z axis lies within a plane of symmetry. If the applied moment is purely $M_y$ or purely $M_z$ (i.e., one or the other is zero) then either the z or the y axis is the neutral axis (as follows from Equ.19). However, even when $I_{yz} = 0$, the neutral axis will not coincide with either the y or the z axis if both $M_y$ and $M_z$ are non-zero, as can be seen from Equ.(19) – and this is obviously to be expected. The neutral axis is then given by,

$$\tan\alpha = \frac{I_z M_y}{I_y M_z} \tag{21}$$

The formula (16) for the stress then reduces to,

$$\sigma = \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{22}$$

which is simply the sum of the stresses due to each moment separately. Alternatively

(20a) reduces to,

$$\sigma = y'\sqrt{\left(\frac{M_z}{I_z}\right)^2 + \left(\frac{M_y}{I_y}\right)^2} \qquad (23)$$

Equs.(22) and (23) are equivalent but (23) has the advantage of showing that to determine the maximum (or minimum) stress it is the maximum (or minimum) of $y'$, the perpendicular distance from the neutral axis, which is required.