Homework: T72S01 Session 7: Formulation of the general elasticity problem

Mentor Guide Knowledge & Skills Question

1.10 Explain the role of displacement compatibility in imposing constraints on the strain fields, and hence state or derive a soluble system of linear elastic differential equations for the general case.

Numerical/Algebraic Question

1) In 2D the basic equations are,

Equilibrium: $\sigma_{x,x} + \tau_{,y} = 0$, $\sigma_{y,y} + \tau_{,x} = 0$ (1)

Hooke's Law: $E\varepsilon_x = \sigma_x - \nu\sigma_y$, $E\varepsilon_y = \sigma_y - \nu\sigma_x$, $G\gamma = \tau$ (2)

Compatibility: $\varepsilon_{x,yy} + \varepsilon_{y,xx} = 2\varepsilon_{xy,xy} = \gamma_{,xy}$ (3)

Definition of Airy function: $\sigma_x = \varphi_{,yy}$ $\sigma_y = \varphi_{,xx}$ $\sigma_{xy} = \tau = -\varphi_{,xy}$ (4)

(a)Show that the equilibrium equations, (1), are a consequence of the existence of an Airy function obeying (4).

(b)By substituting (4) into (2) and then substituting the result into (3), derive the Airy equation,

$$\nabla^4\varphi = 0 \quad (5)$$

2) Given that Airy's equation in 2D polars is,

$$\left[\partial_r^4 + \frac{2}{r}\partial_r^3 - \frac{1}{r^2}\partial_r^2 + \frac{1}{r^3}\partial_r\right]\varphi = 0 \quad (6)$$

Show by substitution that the general solution is,

$$\varphi = A + Br^2 + C\log r + Dr^2\log r \quad (7)$$

3) The stresses are given in terms of the Airy function in 2D polar coordinates by $\sigma_r = \frac{1}{r^2}\varphi_{,\theta\theta} + \frac{1}{r}\varphi_{,r}$ and $\sigma_\theta = \varphi_{,rr}$. By considering the general solution, (7), with $D = 0$, show that the hoop and radial stresses in a thick pipe under internal pressure P_i and external pressure P_o are, at an arbitrary radius r,

$$\sigma_\theta = \frac{P_iR_i^2 - P_oR_o^2}{R_o^2 - R_i^2} + \frac{(P_i - P_o)R_i^2R_o^2}{(R_o^2 - R_i^2)r^2} \quad \text{and} \quad \sigma_r = \frac{P_iR_i^2 - P_oR_o^2}{R_o^2 - R_i^2} - \frac{(P_i - P_o)R_i^2R_o^2}{(R_o^2 - R_i^2)r^2}$$

4) By considering problem (3) for zero internal pressure and in the limit of a very large external radius, show that the stress concentration factor for a hole in an infinite plate subject to an equi-biaxial stress is 2.

5) Optional problem (hard!): Show that the Airy function $\mathrm{Im}\left\{x\log\frac{z-i}{z+i}\right\}$ represents a point load of magnitude 2π in the y-direction.