Deriving the Solution for a Cubic

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Find the solutions of, $az^3 + bz^2 + cz + d = 0$ (1)

Invent a crazy number system, like complex numbers but with three parts instead of two, i.e. with two different 'imaginary' parts denoted by j and k. The multiplication table is,

	1	j	k
1)	1	j	k
j)	j	k	1
k)	k	1	j

(2)

Of course there is no such sensible number system really[1], as Hamilton found out. But we'll not let that stop us. Write,

$$z = x + jy + kv \tag{3}$$

Substitute (3) into (1) and demand that the three parts of the result (the real part and the two different imaginary parts) are individually zero, just as they would have to be if (2) were a sensible number system. This gives the three equations,

$$a\left(x^3 + y^3 + v^3 + 6xyv\right) + b\left(x^2 + 2yv\right) + cx + d = 0 \tag{4}$$

$$3a\left(xv^2 + x^2 y + y^2 v\right) + b\left(v^2 + 2xy\right) + cy = 0 \tag{5}$$

$$3a\left(x^2 v + xy^2 + yv^2\right) + b\left(y^2 + 2xv\right) + cv = 0 \tag{6}$$

Multiplying (5) by v and (6) by y and subtracting or adding the results gives,

$$(3ax + b)\left(v^3 - y^3\right) = 0 \tag{7}$$

$$(3ax + b)\left(v^3 + y^3\right) + 6a\left(x^2 yv + y^2 v^2\right) + 4bxyv + 2cyv = 0 \tag{8}$$

(7) implies that either $v^3 = y^3$ or that,

$$x = -\frac{b}{3a} \tag{9}$$

We choose to follow the consequences of this latter possibility. For a start it simplifies (8) to,

$$3a\left(x^2 yv + y^2 v^2\right) + 2bxyv + cyv = 0 \tag{10}$$

which implies that $v = 0$ or $y = 0$ or,

$$3a\left(x^2 + yv\right) + 2bx + c = 0 \tag{11}$$

Again we choose to follow the consequences of the latter possibility. Substituting for x from (9) this gives,

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[1] Finite dimensional division rings over the reals must have dimension 1, 2, 4 or 8. The only possible division rings of dimension 1, 2 and 4, up to isomorphism, are the reals, the complex numbers and the quaternions. I'm not sure that the same uniqueness can be claimed for dimension 8, though the octonions are the only *alternative* division ring of dimension 8.

$$yv = \frac{b^2 - 3ac}{(3a)^2} \equiv D \qquad (12)$$

(9) and (12) have been derived from (5) and (6) only, so we now need to use (4). Substituting (9) into (4) it becomes,

$$a\left(y^3 + v^3\right) = -\left(ax^3 + bx^2 + cx + d\right) = -aE \qquad (13)$$

where,

$$E = \frac{2b^3 - 9abc + 27a^2 d}{(3a)^3} \qquad (14)$$

Combining (12) and (13) we have,

$$y^6 + Ey^3 + D^3 = 0 \qquad (15)$$

This yields the solutions,

$$y^3 = Y_{\pm} = \left[-E \pm \sqrt{E^2 - 4D^3}\right]/2 \qquad (16)$$

At first sight we seem to have a superfluity of solutions, since there are three cube roots of each of Y_+ and Y_-, apparently making six solutions in all. However, it is readily seen from (16) that $Y_+ Y_- = D^3$, and comparison with (12) shows that we can therefore interpret,

$$y = \{1, \omega, \omega^2\}(Y_+)^{1/3} \qquad \text{and} \qquad v = \{1, \omega^2, \omega\}(Y_-)^{1/3} \qquad (17)$$

where ω is a complex cube-root of unity, $\omega = \exp\{2\pi i/3\} = \left(-1 + \sqrt{3}i\right)/2$. Each solution for *y* must be matched with the corresponding solution for *v* in order for (12) to be respected.

There remains the interpretation of (3), without which we have merely found the solutions in terms of our crazy number system. We now note that setting $j = \omega$ and $k = \omega^2$ respects the multiplication table, (2). Hence, finally we have the three solutions to the general cubic,

$$z = -\frac{b}{3a} + \{\omega, \omega^2, 1\}(Y_+)^{1/3} + \{\omega^2, \omega, 1\}(Y_-)^{1/3} \qquad (18)$$

where $Y_{\pm}$ are given by (16) using (12) and (14) for D and E. **QED**.

Note that we could not simply set $j = \omega$ and $k = \omega^2$ from the start because we would not then have arrived at the three equations (4), (5), (6). Instead, working in ordinary complex numbers, we would have had just two equations. So our ruse of using the crazy number system is to get the extra equation. The fact that this number system ultimately does not make sense does not undermine the solution, (18), since it is now guaranteed that substitution of (18) into the LHS of (1) will reveal it to be identically zero.

Note that there is always one real root, as we know there has to be. If $Y_{\pm}$ are real and different, then there is only one real root, which is the third in the order listed in (18). If $Y_{\pm}$ are real and equal, or if they are complex (and hence conjugates) then there are three real roots.