A Cauchy-like Integral Theorem for Quaternionic and Biquaternionic Gradnull Functions

The paper which accompanies this note on my site is the more elegant and briefer account of these integral theorems. These notes originated in my original derivation of circa 2003 but are rather "sledgehammer". However, they have the advantage carrying out some of the integrals in more detail, and also presenting some introductory material which may be helpful for the beginner. However, the paper is more enlightening as regards the reasons why these integrals are surface independent due to their derivation from the Stokes-Cartan theorem.

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1. Quaternions: Basic Formulation

I begin with an illustration of how quaternions can assist mathematical intuition.

1.1 A Four Squares Theorem

If two integers *x* and *y* have squares which can each be expressed as the sum of four integers squared, then the product *xy* has the same property.

The theorem appears to be one of those difficult theorems in higher arithmetic. And yet it is very simple to prove. But what has it got to do with quaternions? Consider the same theorem with "sum of four integers" replaced with "sum of two integers". This theorem is easy to demonstrate directly, since, if,

$$x^2 = x_1^2 + x_2^2 \text{ and } y^2 = y_1^2 + y_2^2 \text{ then,} \tag{1.1.1}$$

$$\begin{aligned}(xy)^2 &= (x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2 \\ &= (x_1 y_1 - x_2 y_2)^2 + (x_1 y_2 + x_2 y_1)^2\end{aligned} \tag{1.1.2}$$

hence proving the theorem (since the terms inside the brackets must clearly be integers if each of the x_i and y_i are integers). The proof depends on the factorisation in the last step. In the case of four squares, the proof can be written in similar fashion, but the factorisation step is less than obvious. However, we notice something about the two terms in the two-squares theorem. Defining complex numbers,

$$z_x = x_1 + i x_2 \text{ and } \quad z_y = y_1 + i y_2 \tag{1.1.3}$$

we see that the two terms are just the real and imaginary parts of the product $z_x z_y$. Moreover, the original integers x and y are just the moduli of the complex numbers z_x and z_y. Hence the two-squares theorem follows from,

$$x^2 = |z_x|^2 \text{and } y^2 = |z_y|^2 \text{ imply } (xy)^2 = |z_x|^2 |z_y|^2 = |z_x z_y|^2 \qquad (1.1.4)$$

together with,

(a) the multiplication rule for complex numbers, and,

(b) the fact that the square-modulus of a complex number is the sum of the squares of its real and imaginary parts.

Now the relevance of quaternions to the proof of the four-squares theorem becomes clear. We replace (1.1.3) with two quaternions,

$$z_x = x_t + Ix_1 + Jx_2 + Kx_3 \text{ and } z_y = y_t + Iy_1 + Jy_2 + Ky_3 \qquad (1.1.5)$$

Noting that (1.4) remains true for quaternions we see that the four-squares theorem is true and that the factorisation process must result in the four component parts of the product quaternion, $z_x z_y$. We can conclude this without knowing what the explicit expressions for these terms are, nor knowing what the multiplication rule for quaternions actually is. It suffices to know that quaternions, with property (b), above, exist.

Please do not confuse this four squares theorem with Lagrange's four-squares theorem, which is completely different.

It is thus clear that the theorem is also true if "four squares" is replaced with "eight squares" since we know octonions exist for which (1.1.4) and (b) above hold. The theorem is not true for anything other than 2, 4 or 8 squares, a result known as Hurwitz's theorem, 1898, published posthumously in 1923, see Penrose, first edition 2004. This is related to the fact that the only real finite-dimensional, alternative division algebras over the reals are the reals themselves, the complex numbers, the quaternions and the octonians.

1.2 Quaternion Algebra: The Basics

Now to develop quaternion algebra to show they do actually exist. The algebra of quaternions is based on Hamilton's famous,

$$I^2 = J^2 = K^2 = IJK = -1 \qquad (1.2.1)$$

From this it follows that the algebra is non-commutative. Starting from $IJK = -1$ and using the fact that each of I, J, K are square-roots of -1 we find,

1) Pre-multiplying by I gives $JK = I$
2) Pre-multiplying this by J gives $K = -JI$
3) Post-multiplying this by I gives $KI = J$
4) Pre-multiplying this by K gives $I = -KJ$
5) Post-multiplying by J gives $IJ = K$
6) Pre-multiplying this by I gives $J = -IK$

Putting these together shows that I, J, K anti-commute, i.e.,

$$IJ = -JI = K, \qquad JK = -KJ = I \qquad KI = -IK = J \tag{1.2.2}$$

The multiplication table is thus,

	1	I	J	K
1	1	I	J	K
I	I	-1	K	$-J$
J	J	$-K$	-1	I
K	K	J	$-I$	-1

Where the row label indicates the symbol written on the left, e.g., IJ is row I and column J.

A general (real) quaternion is composed of four real components, or coordinates,

$$a = a_t + Ia_x + Ja_y + Ka_z \tag{1.2.3}$$

By a real quaternion we shall mean that a_t, a_x, a_y, a_z are real. We will later, in section ?, consider the numbers a_t, a_x, a_y, a_z to be complex, thus defining biquaternions. Until then, all quaternions are real.

To distinguish the terms 'real' and 'imaginary' as applied to the usual 2D complex numbers, the quaternion parts will be referred to as the 'temporal' or 'scalar' part (a_t), and the 'spatial' or 'vector' parts, (a_x, a_y, a_z).

Considering I, J, K to be unit vectors along the x, y, z directions, and denoting a 3-vector by an over-scored bar, e.g. $\bar{r}$, a general (real) quaternion can be written more compactly as,

$$a = a_t + \bar{a} \tag{1.2.4}$$

If two quaternions have zero temporal parts, $a_t = b_t = 0$, then their product is,

$$ab = \left(Ia_x + Ja_y + Ka_z\right)\left(Ib_x + Jb_y + Kb_z\right) \tag{1.2.5}$$
$$= -\left(a_x b_x + a_y b_y + a_z b_z\right) + I\left(a_y b_z - a_z b_y\right) + J(a_z b_x - a_x b_z) + K\left(a_x b_y - a_y b_x\right)$$

In vector notation this is simply,

$$ab = -\bar{a} \cdot \bar{b} + \bar{a} \times \bar{b} \qquad (\text{for } a_t = b_t = 0) \tag{1.2.6}$$

The RHS of (1.2.6) is nonsensical in usual vector notation since it asks us to 'add' a scalar to a vector. It makes sense as a quaternion because the 'scalar' part is the temporal part, and the vector part is the quaternion spatial part which depends upon the I, J, K. Note that the anti-commuting nature of the 3-vector cross product may be considered to arise from that of I, J, K.

Note that for a quaternion with zero temporal part, (1.2.4) reduces to $a = \bar{a}$. Using this notation in (1.2.6) gives,

$$\bar{a}\bar{b} = -\bar{a} \cdot \bar{b} + \bar{a} \times \bar{b} \tag{1.2.7}$$

where a simple juxtaposition of two 3-vectors (understood to be quaternions) represents the quaternionic product. The quaternion product of two 3-vectors is seen to break down into a combination of the usual dot and cross products of 3-vectors.

The product of two general quaternions, with all four coordinates non-zero, can now be found simply, as follows,

$$ab = (a_t + \bar{a})(b_t + \bar{b}) = a_t b_t + a_t \bar{b} + b_t \bar{a} + \bar{a}\bar{b}$$
$$= (a_t b_t - \bar{a} \cdot \bar{b}) + (a_t \bar{b} + b_t \bar{a} + \bar{a} \times \bar{b}) \quad (1.2.8)$$

where the two bracketed expressions are the temporal and vector parts of the product quaternion respectively. It is curious that the temporal part of the product is just the Minkowskian 4-vector inner product if a_t, a_x, a_y, a_z are interpreted as components of a 4-vector. The minus sign in the Minkowski metric arises from the fact that I, J, K are square-roots of unity.

Just as the anti-commuting nature of the 3-vector cross product can be considered to arise from that of I, J, K, so the reverse could be claimed: that the non-commuting nature of quaternions is due to the presence of the 3-vector cross product in (1.2.8). It follows from (1.2.8) that,

$$ab - ba = 2\bar{a} \times \bar{b} \quad (1.2.9)$$

The quaternion-conjugate is defined by,

$$a^{\#} = a_t - Ia_x - Ja_y - Ka_z = a_t - \bar{a} \quad (1.2.10)$$

Note that it follows from (1.2.8) that, for any quaternions a and b,

$$(ab)^{\#} \equiv b^{\#} a^{\#} \quad (1.2.11)$$

the reversal of order being required to ensure the cross-product part of (1.2.8) reverses sign. Note that the use of the symbol $^{\#}$ to denote the quaternionic conjugate is to distinguish it from the complex conjugate * which will later be used as well. It follows from (1.2.8) that,

$$aa^{\#} = a^{\#}a = a_t^2 + |\bar{a}|^2 = a_t^2 + a_x^2 + a_y^2 + a_z^2 \quad (1.2.12)$$

the vector part being identically zero. (NB: this follows from the fact that $\bar{a} \times \bar{a} \equiv 0$). This is known as the (squared) modulus.

This completes the proof of the four-squares theorem! To find the explicit factorisation we substitute into the quaternion identity,

$$|z_x|^2 |z_y|^2 = |z_x z_y|^2 \quad (1.2.13)$$

Giving,

$$(x_t^2 + x_x^2 + x_y^2 + x_z^2)(y_t^2 + y_x^2 + y_y^2 + y_z^2)$$
$$= (x_t y_t - \bar{x} \cdot \bar{y})^2 + |x_t \bar{y} + y_t \bar{x} + \bar{x} \times \bar{y}|^2$$
$$= (x_t y_t - x_x y_x - x_y y_y - x_z y_z)^2 + (x_t y_x + y_t x_x + x_y y_z - x_z y_y)^2 +$$
$$(x_t y_y + y_t x_y + x_z y_x - x_x y_z)^2 + (x_t y_z + y_t x_z + x_x y_y - x_y y_x)^2$$

This factorisation is not at all obvious, but the proof did not depend upon finding it.

1.3 Generalisation of de Moivre's Theorem

We write a unit vector as $\hat{n}$, i.e., $\hat{n} = In_x + Jn_y + Kn_z$ where $n_x^2 + n_y^2 + n_z^2 = 1$. In terms of the quaternion product, (1.2.8), a unit vector is a square-root of unity, i.e., $\hat{n}\hat{n} = -1$. Hence, if we define the exponential of a vector by the usual power series we get the usual result in terms of trignometric functions, i.e.,

$$e^{\theta \hat{n}} = \sum_{r=0}^{\infty} \frac{(\theta \hat{n})^r}{r!} = cos(\theta) + \hat{n}\, sin(\theta) \quad (1.3.1)$$

From this it follows that the quaternion conjugate is obtained by reversing the sign of either $\hat{n}$ or θ. The product of the above with its conjugate yields unity, confirming that its modulus is unity.

We postulate that an arbitrary quaternion can be expressed as,

$$z = re^{\theta\hat{n}} \tag{1.3.2}$$

Clearly, specifying $r, \hat{n}$ and θ uniquely defines a quaternion, i.e., uniquely defines the four (real) components of z. Is the representation of a given quaternion in the form of (1.3.2) unique? Well, the representation of a complex number in the form $z = e^{i\theta}$ is not unique because an arbitrary integer multiple of 2π can be added to θ and leave z unchanged. A similar situation prevails with (1.3.2), but the representation of a given quaternion can be made unique provided that $r, \hat{n}$ and θ are restricted to a suitable set of ranges. We have,

$$z_t = r\,cos(\theta) \qquad \text{and} \qquad \bar{z} = \hat{n}r\,sin(\theta) \tag{1.3.3}$$

By analogy with spherical polar coordinates we see that all possible vectors, $\bar{z}$, are uniquely specified if we restrict $r\,sin(\theta)$ to be positive but allow the unit vector $\hat{n}$ to range over all eight octants, i.e., to have positive or negative components. If we restrict r to be positive, so that r is the modulus of z, as (1.3.2) suggests, then $sin(\theta)$ must be positive. Restricting θ to the range 0 to π/2 would cover all possible positive values of $sin(\theta)$. However, $cos(\theta)$ is also positive over this range and this would restrict z_t to positive values. To include negative z_t values we extend the range of θ to [0, π]. Hence all quaternions are uniquely represented in the form (1.3.2) if,

$$r \geq 0,\ 0 \leq \theta < \pi \tag{1.3.4}$$

whilst allowing the unit vector $\hat{n}$ to range over all eight octants.

Note that the implication of this is that every vector, $\bar{z}$, occurs twice in the range (1.3.4), once for $0 \leq \theta < \pi/2$ with positive z_t and once for $\pi/2 \leq \theta < \pi$ with equal and opposite (negative) z_t.

This will not preclude our using expressions like (1.3.2) with negative θ, or with θ greater than π. It merely means that the same quaternion could be expressed with θ in the range 0 to π and, if necessary, with the sign of the unit vector $\hat{n}$ reversed.

Note that it follows from (1.3.1) that the reciprocal of a quaternion with unit modulus is obtained by reversing the sign of the exponent, i.e.,

$$e^{\hat{n}\theta}e^{-\hat{n}\theta} \equiv 1 \tag{1.3.5}$$

NB: This is not obvious in view of (1.4.2), below.

1.4 Quaternions as Rotation Operators

Complex numbers may be considered as 2D vectors. Multiplication by a complex number of unit modulus, say $e^{i\varphi}$, represents a rotation in the plane by ϕ. This follows simply from,

$$e^{i\varphi}\left(re^{i\theta}\right) \equiv re^{i(\theta+\varphi)} \tag{1.4.1}$$

Hamilton developed quaternions specifically to provide an algebra of 3-dimensional rotations in analogy with the above. However, for quaternions, the relevant algebraic form cannot be like (1.4.1). For one thing, the non-commutation of quaternions prevents a product of two exponentials being expressed as the exponential of the sum of their exponents, that is, in general for two quaternions a and b,

$$e^a e^b \neq e^{a+b} \text{ in general} \tag{1.4.2}$$

A simple counter example proving (1.4.2) is as follows: consider $a = I\theta$ and $b = J\varphi$. From (1.3.1) we have,

$$LHS = (cos(\theta) + I\,sin(\theta))(cos(\varphi) + J\,sin(\varphi))$$

$$RHS = cos(\alpha) + \hat{n}\,sin(\alpha), \text{ where } \alpha = \sqrt{\theta^2 + \varphi^2},\ \hat{n} = (I\theta + J\varphi)/\alpha$$

Hence it is clear that the RHS has no K-component, whereas a K-component does arise in the LHS from the cross-product, namely $sin(\theta)sin(\varphi)$. Hence, the two quaternions are distinct unless either θ or φ is zero.

So, why should quaternions have anything to do with rotations? Consider two quaternions expressed in the standard form of (1.3.2). This shows that the modulus of their product is the product of their moduli, since,

$$ab = \left(re^{\hat{n}\theta}\right)\left(se^{\hat{m}\varphi}\right) = rse^{\hat{n}\theta}e^{\hat{m}\varphi}$$

and hence

$$|ab|^2 = r^2 s^2 e^{-\hat{m}\varphi} e^{-\hat{n}\theta} e^{\hat{n}\theta} e^{\hat{m}\varphi} = r^2 s^2 \tag{1.4.3}$$

where we have made use of (1.2.11) and (1.3.5). Consequently, if we choose one quaternion (say b) to have unity modulus ($s = 1$) then multiplying a by b preserves the modulus of a. Since, by (1.2.12), the (squared) modulus is just the sum of the squares of the four components of the quaternion a, the operation of multiplying by a unit quaternion represents a rotation in the 4-dimensional space defined by all four components of a.

This is rather more than we bargained for!

Note, however, that simple multiplication by a unit quaternion cannot generate an ***arbitrary*** rotation in 4-dimensions. The latter would require 6 free parameters (in relativistic terminology, 3 for the possible rotations of 3-space plus 3 for the boosts). In contrast, a unit quaternion has only 3 free parameters.

Before developing the general 3-space rotation in terms of quaternions, we investigate the action of simple pre-multiplication by I. For complex numbers this would be a rotation by π/2. For an arbitrary quaternion we have,

$$Ia = I(a_t + Ia_x + Ja_y + Ka_z) = (-a_x + Ia_t) + (-Ja_z + Ka_y) \tag{1.4.4}$$

The brackets help to recognise that this represents a rotation by π/2 in both the (t, x) plane and the (y, z) plane simultaneously. Post-multiplying (1.4.4) by $-I$ gives,

$$Ia(-I) = (a_t + Ia_x) + (-Ja_y - Ka_z) \tag{1.4.5}$$

which represents a rotation of 3-space about the x-axis by π, whilst the (t, x) plane is unchanged. This suggests that a general rotation of 3-space about an axis $\hat{m}$ by an angle θ may be of the form,

$$a \to a' = e^{\hat{m}\theta/2} a e^{-\hat{m}\theta/2} \tag{1.4.6}$$

It is clear that this transformation does not change the temporal component, since this commutes with the quaternion factors, which then multiply to unity by (1.3.5). As regards the vector part, it suffices to consider $a = I$. If the action of the transformation (1.4.6) is to rotate I about $\hat{m}$ by θ then the same will be true for J and K (by symmetry), and hence true also for any vector by linear superposition. The proof for $a = I$ is as follows,

$$I\hat{m} = -m_x + Km_y - Jm_z \text{ and } \hat{m}I = -m_x - Km_y + Jm_z \tag{1.4.7}$$

$$[I, \hat{m}] = 2(Km_y - Jm_z) \tag{1.4.8}$$

$$\hat{m}i\hat{m} = i - 2m_x\hat{m} \quad \text{(by expansion \& simplification)} \tag{1.4.9}$$

Hence (1.4.6) with $a = I$ (1.4.6) becomes, (1.4.10)

$$\begin{aligned}
&e^{\hat{m}\theta/2}Ie^{-\hat{m}\theta/2} = (cos(\theta/2) + \hat{m}\,sin(\theta/2))I(cos(\theta/2) - \hat{m}\,sin(\theta/2))\\
&= I\,cos^2(\theta/2) - [I, \hat{m}]\,sin(\theta/2)\,cos(\theta/2) - \hat{m}I\hat{m}\,sin^2(\theta/2)\\
&= I\,cos^2(\theta/2) - 2(Km_y - Jm_z)\,sin(\theta/2)\,cos(\theta/2) - (I - 2m_x\hat{m})\,sin^2(\theta/2)\\
&= I[cos^2(\theta/2) - sin^2(\theta/2) + 2m_x^2\,sin^2(\theta/2)]\\
&+J[2m_z\,sin(\theta/2)\,cos(\theta/2) + 2m_xm_y\,sin^2(\theta/2)]\\
&+K[-2m_y\,sin(\theta/2)\,cos(\theta/2) + 2m_xm_z\,sin^2(\theta/2)]\\
&= I[cos(\theta) + m_x^2(1 - cos(\theta))]\\
&+J[m_z\,sin(\theta) + m_xm_y(1 - cos(\theta))]\\
&+K[-m_y\,sin(\theta) + m_xm_z(1 - cos(\theta))]
\end{aligned}$$

That this does indeed give the correct I, J, K components for the unit x-vector rotated by an angle θ about the axis $\hat{m}$ can be confirmed by standard vector analysis (see Appendix A for details).

Hence we conclude that (1.4.6) also provides the most general rotation of a vector in 3-space in quaternion notation. (For an alternative, rather more elegant proof, see *Five Square Roots*, Appendix L, §L.2).

2. Differentiation of Quaternionic Functions

2.1 A False Start

For ordinary complex numbers we will use $i = \sqrt{-1}$, not to be confused with the quaternion I, although that is also a square root of -1.

For such complex numbers, a function $f(z)$ is holomorphic in the neighbourhood of a point $z = x + iy$ if its derivative has a unique value within that neighbourhood irrespective of the direction in the Argand plane from which the limiting process $\delta z \to 0$ is taken. This leads immediately to the Cauchy-Riemann equations and the fact that holomorphic functions are harmonic, i.e.,

$$\frac{df}{dz} = \frac{df}{dx} = \frac{df}{idy} \Rightarrow f_{r,x} = f_{i,y} \text{ and } f_{r,y} = -f_{i,x} \Rightarrow (\partial_x^2 + \partial_y^2)f = 0 \tag{2.1.1}$$

where subscripts r and i represent real and imaginary parts of f.

For complex functions, being holomorphic is sufficient to show that all the higher derivatives exists within the same neighbourhood, and hence that the Taylor series exists – so that complex holomorphic functions are also analytic functions. (See any standard text or *Five Square Roots* for details).

For quaternions we note that expressions like $\frac{df}{dz}$ are ambiguous because this could be interpreted either as $\frac{1}{dz}.df$ or as $df.\frac{1}{dz}$, which are generally different because quaternions do not commute. We adopt the following conventions,

$$\frac{a}{b} \equiv \frac{1}{b}.a \qquad \text{but} \qquad {}^{a}/_{b} \equiv a.\frac{1}{b} \tag{2.1.2}$$

In what follows we shall virtually always use the first form.

If we attempt the deploy the same definition of 'holomorphic' for a quaternion-valued function of a quaternionic variable, $q = t + Ix + Jy + Kz$, namely that the derivative exists and is the same irrespective of the direction of the limiting process in 4D quaternion space, then what results is four quaternion expressions which must all be equal,

$$\begin{aligned}\frac{df}{dq} = \frac{df}{dt} &= \frac{df_0}{dt} + I\frac{df_1}{dt} + J\frac{df_2}{dt} + K\frac{df_3}{dt} \\ = \frac{df}{Idx} &= -I\frac{df_0}{dx} + \frac{df_1}{dx} - K\frac{df_2}{dx} + J\frac{df_3}{dx} \\ = \frac{df}{Jdy} &= -J\frac{df_0}{dy} + K\frac{df_1}{dy} + \frac{df_2}{dy} - I\frac{df_3}{dy} \\ = \frac{df}{Kdz} &= -K\frac{df_0}{dz} - J\frac{df_1}{dz} + I\frac{df_2}{dz} + \frac{df_3}{dz}\end{aligned} \tag{2.1.3}$$

which gives,

$$\frac{df_0}{dt} = \frac{df_1}{dx} = \frac{df_2}{dy} = \frac{df_3}{dz} \text{ and } \frac{df_1}{dt} = -\frac{df_0}{dx} = -\frac{df_3}{dy} = \frac{df_2}{dz} \text{ and}$$
$$\frac{df_2}{dt} = \frac{df_3}{dx} = -\frac{df_0}{dy} = -\frac{df_1}{dz} \text{ and } \frac{df_3}{dt} = -\frac{df_2}{dx} = \frac{df_1}{dy} = -\frac{df_0}{dz} \tag{2.1.4}$$

These equations imply that all components of f satisfy all possible 2D Laplace equations, that is $(\partial_a^2 + \partial_b^2)f_c = 0$ where a, b, c represent any of the four components (subject to $a \neq b$). This appears to be far too restricted a class of quaternionic functions to have great utility. This approach is not considered further.

2.2 The 'Zero Gradient' Approach

An alternative view of the definition of holomorphic complex functions is as follows: consider a gradient operator in the Argand plane, $d = \partial_x + i\partial_y$. A function is then holomorphic (and hence analytic) in the neighbourhood of a point $z = x + iy$ if the "gradient" is zero, i.e., $df(z) = 0$, within that neighbourhood. This is seen to be equivalent to the Cauchy-Riemann conditions, (2.1.1). Note that the 2D Laplace equation follows elegantly from this definition simply by multiplying $df(z) = 0$ by $d^* = \partial_x - i\partial_y$ and noting that $d^*d = \partial_x^2 + \partial_y^2$.

Hence we use this approach as an appropriate means of defining an interesting class of quaternion-valued functions of the quaternionic variable $q = t + Ix + Jy + Kz$. Defining the gradient operator in quaternion space by analogy we have,

$$d = \partial_t + \bar{\nabla}, \text{ where } \bar{\nabla} = I\partial_x + J\partial_y + K\partial_z$$

and hence, $\bar{\nabla}\bar{\nabla} = -\bar{\nabla} \cdot \bar{\nabla} = -\nabla^2, \ \nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$

$$d^{\#}d = \partial_t^2 + \nabla^2 \tag{2.2.1}$$

where $d^\# = \partial_t - \bar{\nabla}$. Note also that $d^2 = \partial_t^2 - \nabla^2 + 2\partial_t\bar{\nabla}$ so that neither d^2 nor $d^\# d$ give us the wave operator in real (Minkowski) spacetime (which, of course, for a scalar field is $(\partial_t^2 - \nabla^2)\psi = 0$ in units in which the speed of light is 1). To represent real (Minkowski) spacetime will oblige us to consider biquaternions later – but for now we shall stick to real quaternions.

We now define a class of quaternion-value functions, $f(t,x,y,z)$, by the vanishing of their quaternionic "gradient", at least within a specified region,

$$df(t,x,y,z) = 0 \tag{2.2.2}$$

We define a quaternionic-valued function of the four real, scalar variables t,x,y,z as being "gradnull" within a specified region if (2.1.6) holds within that region.

Note that the definition of gradnull does require the four derivatives wrt the four real variables t,x,y,z to exist.

Note that we do NOT write such functions as $f(q)$, as if they were functions of the quaternionic variable $q = t + Ix + Jy + Kz$. Instead, gradnull functions depend separately on the four real variables t,x,y,z – except that the dependence of the function on these variables is constrained by (2.1.6). In fact, gradnull functions cannot be a function of q only because in that case we would have,

$$df(q) = f'(q)\left[\frac{\partial q}{\partial t} + I\frac{\partial q}{\partial x} + J\frac{\partial q}{\partial y} + K\frac{\partial q}{\partial z}\right] = f'(q)[1 + I^2 + J^2 + K^2] = -2f'(q) \neq 0$$

the only exception being the trivial case that f is a constant.

(2.1.6) immediately implies that quaternionic gradnull functions obey Laplace's equation in 4D Euclidean space,

$$d^\# df = (\partial_t^2 + \nabla^2)f = 0 \tag{2.2.3}$$

This differs from the wave equation since all terms enter with a positive sign.

We can also define conjugate-gradnull functions by,

$$d^\# f(t,x,y,z) = 0 \tag{2.2.4}$$

These also obey the 4D Euclidean Laplace equation because,

$$dd^\# f = (\partial_t^2 + \nabla^2)f = 0 \tag{2.2.5}$$

Considering an arbitrary quaternion-valued function of the spatial coordinates x,y,z only, $g(x,y,z)$, it is clear that the function,

$$f(t,x,y,z) = e^{-t\bar{\nabla}}g(x,y,z) \tag{2.2.6}$$

is gradnull because,

$$d\left(e^{-t\bar{\nabla}}g\right) = (\partial_t + \bar{\nabla})e^{-t\bar{\nabla}}g = (-\bar{\nabla} + \bar{\nabla})e^{-t\bar{\nabla}}g \equiv 0 \tag{2.2.7}$$

Similarly, considering an arbitrary quaternion-valued function of the spatial coordinates x,y,z only, $h(x,y,z)$, it is clear that the function,

$$f(t,x,y,z) = e^{t\bar{\nabla}}h(x,y,z) \tag{2.2.8}$$

is conjugate-gradnull because,

$$d^\#\left(e^{t\bar{\nabla}}h\right) = (\partial_t - \bar{\nabla})e^{t\bar{\nabla}}h = (\bar{\nabla} - \bar{\nabla})e^{t\bar{\nabla}}h \equiv 0 \tag{2.2.9}$$

It is clear that any function f of the form (2.1.10) is gradnull. The reverse also follows is we assume the Taylor series in t exists. That is, any gradnull function can be written like (2.1.10), namely as $f(t,x,y,z) = e^{-t\bar{\nabla}} f(0,x,y,z)$, if all t-derivatives of f exist so we can write,

$$\partial_t f = -\bar{\nabla} f = -\bar{\nabla} f_0 - t\bar{\nabla} f_0' - \frac{t^2}{2}\bar{\nabla} f_0'' - \frac{t^3}{3!}\bar{\nabla} f_0''' - \cdots \qquad (2.2.10)$$

where the subscripts 0 denote evaluation at $t = 0$ and the dashes denote derivatives wrt t. Integration of (2.1.14) then gives $f(t,x,y,z) = e^{-t\bar{\nabla}} f(0,x,y,z)$, QED. Similarly, all conjugate-gradnull functions can be written like (2.1.12) if the time Taylor series exists.

Moreover, Hamilton (1858), following Graves, showed that these functions exhaust all possible solutions to the 4D Euclidean Laplace equation, the most general solution being a sum of the two types,

$$(\partial_t^2 + \nabla^2) f = 0 \Leftrightarrow f = e^{-t\bar{\nabla}} g(x,y,z) + e^{t\bar{\nabla}} h(x,y,z) \qquad (2.2.11)$$

where g and h are arbitrary quaternion-valued functions of the spatial coordinates only. Hence gradnull and conjugate-gradnull functions are disjoint classes of particular solutions of Laplace's equation in 4D Euclidean space.

2.3 General Form of Gradnull Quaternionic Functions

We already know the answer is (2.1.10). But is there a nearest-equivalent of holomorphic complex functions being a function of $z = x + iy$ only? There is, but it is far less useful and will play no part in our subsequent development. The Graves-Hamilton theorem states that, for gradnull quaternionic functions, i.e., functions that can be written as (2.1.10), we can write

$$df = 0 \Leftrightarrow f = e^{-t\bar{\nabla}} g(x,y,z) \equiv Mg(x - It, y - Jt, z - Kt) \qquad (2.3.1)$$

Where the "mean value", denoted M, refers to the mean being taken over all possible orderings of the non-commuting I, J, K factors in the Taylor series expansion of g.

In the case of holomorphic complex functions there is no lack of commutation, and the "mean value" may be dropped, so for these we have,

$$(\partial_x + i\partial_y) f = 0 \Leftrightarrow f = e^{-ix\partial_y} g(y) \equiv g(y - ix) \equiv \tilde{g}(x + iy) \equiv g(z)$$

as is well known.

For gradnull quaternion functions, the nearest generalisation of this is (2.3.1). Were it not for the "mean value" part, (2.3.1) would specify a function which is separately dependent upon the three complex variables $t + Ix$, $t + Jy$ and $t + Kz$. This is consistent with our observation, above, that a gradnull quaternion function is not simply a function of the quaternion variable $q = t + Ix + Jy + Kz$ but more general an less restricted.

2.4 The Zero-Gradient Condition in Vector Notation

Recall that quaternion multiplication gives, from (1.2.8),

$$ab = (a_t b_t - \bar{a} \cdot \bar{b}) + (a_t \bar{b} + \bar{a} b_t + \bar{a} \times \bar{b})$$

(The brackets merely serve to emphasise the temporal and 3-vector parts). Consequently, we may expand the quaternion gradient of a function f as,

$$df = (\partial_t + \bar{\nabla})(f_0 + \bar{f}) \equiv (\partial_t f_0 - \bar{\nabla}\cdot\bar{f}) + (\partial_t \bar{f} + \bar{\nabla} f_0 + \bar{\nabla}\times\bar{f}) \qquad (2.4.1)$$

Hence, all three basic vector derivatives occur in df; the grad, the div and the curl, as well as the 'time' derivative. The gradnull condition, $df = 0$, is thus re-written as,

$$(\partial_t f_0 - \bar{\nabla}\cdot\bar{f}) = 0 \text{ and } (\partial_t \bar{f} + \bar{\nabla} f_0 + \bar{\nabla}\times\bar{f}) = \bar{0} \qquad (2.4.2)$$

We know that $df = 0$ trivially implies that the 4D Laplace equation is obeyed by all four components of f. However, this is not immediately obvious from the vector expressions for $df = 0$, (2.4.2). The Laplace equation for f_0 is fairly obvious, in that taking the div of the second equation gives,

$$\partial_t \bar{\nabla}\cdot\bar{f} + \nabla^2 f = 0 \qquad (2.4.3)$$

because div(curl…) is identically zero. It is clear from the time derivative of the first equation in (2.4.2) that the first term in (2.4.3) is just the second time derivative of f_0, and hence (2.4.3) gives the 4D Laplace equation for f_0, i.e. $(\partial_t^2 + \nabla^2) f_0 = 0$. We have to work harder to derive the Laplace equations for the vector components from (2.4.2). Take the curl of the second equation to give,

$$\partial_t \bar{\nabla}\times\bar{f} + \bar{\nabla}\times(\bar{\nabla}\times\bar{f}) = 0$$

because curl (grad(…)) is zero. Then using the standard expression for curl(curl(…))

$$\partial_t \bar{\nabla}\times\bar{f} + \bar{\nabla}(\bar{\nabla}\cdot\bar{f}) - \nabla^2\bar{f} = 0$$

Using both equations in (2.4.2) gives,

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{f}) = \bar{\nabla}\partial_t f_0 = -\partial_t^2\bar{f} - \partial_t\bar{\nabla}\times\bar{f} \Rightarrow \bar{\nabla}(\bar{\nabla}\cdot\bar{f}) + \partial_t\bar{\nabla}\times\bar{f} = -\partial_t^2\bar{f}$$

which, when substituted in the preceding equation, gives the 4D Laplace equations for the vector parts, i.e. $(\partial_t^2 + \nabla^2)\bar{f} = 0$ as required.

Hence, whilst the derivation of the Laplace equations is immediate and trivial from the quaternion form $df = 0$, it is lengthy and not at all obvious from the vector form, (2.4.2). This would appear to illustrate quite dramatically the power of quaternion notation over vector notation.

3. Explicit Evaluation of Derivatives of Quaternionic Functions

3.1 Quaternion 'Grad' of Powers of the Configuration Quaternion, dq^n

The 'configuration quaternion', q, is the position 'vector' in 4D quaternion space, i.e.,

$$q = t + \bar{r} \qquad (3.1.1)$$

It is surprising to discover that $dq = -2$, as follows,

$$dq = (\partial_t + \bar{\nabla})(t + \bar{r}) = 1 + \bar{\nabla}\bar{r} = 1 - \bar{\nabla}\cdot\bar{r} = 1 - 3 = -2 \qquad (3.1.2)$$

If we incorrectly assumed the usual chain-rule applied, one would expect that dq^2 would be $2q dq = -4q$, but this is not the case, as we shall now see. Firstly,

$$q^2 = (t^2 - r^2) + 2t\bar{r} \qquad (3.1.3)$$

using the quaternion multiplication rule, (1.2.8). The same rule allows the 'grad', df, to be expressed as in (2.4.1), giving,

$$dq^2 = \partial_t(t^2 - r^2) - \bar{\nabla}\cdot(2t\bar{r}) + \partial_t(2t\bar{r}) + \bar{\nabla}(t^2 - r^2) + \bar{\nabla}\times(2t\bar{r}) \qquad (3.1.4)$$

The last term is zero since $\bar{\nabla}$ commutes with t, and $\bar{\nabla} \times \bar{r} \equiv 0$. The remaining terms are evaluated using,

$$\bar{\nabla} \cdot \bar{r} = 3 \quad \text{and} \quad \bar{\nabla} r^2 = 2\bar{r} \tag{3.1.5}$$

giving,

$$dq^2 = 2t - 6t + 2\bar{r} - 2\bar{r} = -4t \tag{3.1.6}$$

The chain rule in its simplest form fails because q and d do not commute. A correct form of chain rule will be derived shortly. We may also consider the configuration quaternion translated by a fixed amount, i.e., replacing q by q – q_0, where $q_0 = t_0 + \bar{r}_0$. We get, for example,

$$d(q - q_0) = -2 \quad \text{and} \quad d(q - q_0)^2 = -4(t - t_0) \tag{3.1.7}$$

The second expression holds despite the usual chain rule not being applicable.

Next consider dq^3. We find,

$$q^3 = (t^2 - 3r^2)t + (3t^2 - r^2)\bar{r} \tag{3.1.8}$$

Hence it is straightforward to evaluate,

$$dq^3 = -6t^2 + 2r^2 \quad \text{and} \quad d(q - q_0)^3 = -6(t - t_0)^2 + 2|\bar{r} - \bar{r}_0|^2 \tag{3.1.9}$$

where we have made use of,

$$\bar{\nabla} \times (r^2\bar{r}) = 0 \text{and} \quad \bar{\nabla} \cdot (r^2\bar{r}) = 5r^2 \tag{3.1.10}$$

as well as (3.15). No clear pattern is emerging as yet from these results, so we now investigate chain rules for quaternion differentiation.

3.2 Chain Rules for Quaternion Differentiation

Because d is a quaternionic operator it does not in general commute with a quaternion – even a quaternion constant. Hence, for arbitrary quaternionic functions F and G,

$$d(FG) \neq (dF)G + F(dG) \quad \text{in general} \tag{3.2.1}$$

However, the ordinary chain rule of differentiation still applies. Hence it is clear that we may write,

$$d(FG) \equiv (dF)G + d(\{F\}G) \tag{3.2.2}$$

where items within {…} are treated as constants as far as differentiation is concerned. In the second term of (3.2.2) the order of the quaternion products is preserved, but the differential operator acts on G only. A means of putting the {…} condition into effect is by re-writing the operator as,

$$d\{F\} \equiv (F^{\#}d^{\#})^{\#} \tag{3.2.3}$$

The point here is that, for arbitrary quaternions (not operators) we have $(b^{\#}a^{\#})^{\#} \equiv ab$, so (3.2.3) gets the quaternion product right, but at the same time puts F to the left of the differential operator so that it is not acted upon by it – just as is required. Hence we get Chain Rule No.1:-

$$d(FG) = (dF)G + (F^{\#}d^{\#})^{\#}G \tag{3.2.4}$$

This is not merely empty notation. It does provide an alternative algorithm for evaluating the LHS. Its correctness can be checked by expanding both sides in terms of components (and is rather tedious). As an illustration of the use of Chain Rule No.1, consider dq^2. We identify F = G = q. Hence,

$$F^{\#}d^{\#} \equiv (t-\bar{r})(\partial_t - \bar{\nabla}) = (t\partial_t - \bar{r}\cdot\bar{\nabla}) - \bar{r}\partial_t - t\bar{\nabla} + \bar{r}\times\bar{\nabla}$$

Hence, (3.2.4) gives,

$$dq^2 = (dq)q + [(t\partial_t - \bar{r}\cdot\bar{\nabla}) + \bar{r}\partial_t + t\bar{\nabla} - \bar{r}\times\bar{\nabla}]q \qquad (3.2.5)$$

Evaluating (3.2.5) gives,

$$\begin{aligned} dq^2 &= -2q + (t\partial_t - \bar{r}\cdot\bar{\nabla})t - (\bar{r}\partial_t + t\bar{\nabla} - \bar{r}\times\bar{\nabla})\cdot\bar{r} + (t\partial_t - \bar{r}\cdot\bar{\nabla})\bar{r} \\ &\quad + (\bar{r}\partial_t + t\bar{\nabla} - \bar{r}\times\bar{\nabla})t + (\bar{r}\partial_t + t\bar{\nabla} - \bar{r}\times\bar{\nabla})\times\bar{r} \\ &= -2q + t - 3t - \bar{r} + \bar{r} + 2\bar{r} \\ &= -4t - 2\bar{r} + 2\bar{r} \\ &= -4t \end{aligned} \qquad (3.2.6)$$

which is the correct answer (see 3.1.6). However, this method was hardly simpler! To evaluate (3.2.6) required,

$$(\bar{r}\times\bar{\nabla})\cdot\bar{r} \equiv 0 \text{ and } (\bar{r}\cdot\bar{\nabla})\bar{r} \equiv \bar{r} \text{ and } (\bar{r}\times\bar{\nabla})\times\bar{r} \equiv -2\bar{r} \qquad (3.2.7)$$

So the application of Chain Rule No.1 appears, at least in this example, to be far more complicated than a direct evaluation of $d(FG)$. Nevertheless, the chain rule is correct.

A second chain rule is of greater utility. To write this rule compactly we introduce a notation $^{\#I}$ which forms the conjugate with respect to I only. Thus,

$$\begin{aligned} a^{\#I} &\equiv a_0 - Ia_1 + Ja_2 + Ka_3 \\ a^{\#J} &\equiv a_0 + Ia_1 - Ja_2 + Ka_3 \\ a^{\#K} &\equiv a_0 + Ia_1 + Ja_2 - Ka_3 \end{aligned} \qquad (3.2.8)$$

This may be combined with the full conjugate, for example,

$$a^{\#\#I} \equiv a_0 + Ia_1 - Ja_2 - Ka_3 \qquad (3.2.9)$$

<u>Chain Rule No.2</u> can then be written,

$$d(FG) = (dF)G + f_0 dG + If_1 d^{\#\#I}G + Jf_2 d^{\#\#J}G + Kf_3 d^{\#\#K}G \qquad (3.2.10)$$

where f_μ are the components of F. This rule may be established as follows,

$$\begin{aligned} d(FG) &= (dF)G + d(\{F\}G) \\ &= (dF)G + F\,\partial_t G + IF\partial_x G + JF\partial_y G + KF\partial_z G \\ &= (dF)G + (f_0 + If_1 + Jf_2 + Kf_3)\partial_t G \\ &\quad + (f_0 + If_1 - Jf_2 - Kf_3)I\partial_x G \\ &\quad + (f_0 - If_1 + Jf_2 - Kf_3)J\partial_y G \\ &\quad + (f_0 - If_1 - Jf_2 + Kf_3)K\partial_z G \\ &= (\mathrm{d}F)G + f_0(\partial_t G + I\partial_x G + J\partial_y G + K\partial_z G) \\ &\quad + If_1(\partial_t G + I\partial_x G - J\partial_y G - K\partial_z G) \\ &\quad + Jf_2(\partial_t G - I\partial_x G + J\partial_y G - K\partial_z G) \\ &\quad + Kf_3(\partial_t G - I\partial_x G - J\partial_y G + K\partial_z G) \end{aligned} \qquad (3.2.11)$$

which is just (3.2.10) written out in full, and where we have used the fact that I anti-commutes with J and K, etc.

Chain Rule No.2, (3.2.10), is a much more useful rule. To illustrate its use we consider $F = q^{n-1}$ and $G = q$ as a means of evaluating the derivative of q^n. We recall that $dq = -2$, and we also find,

$$d^{\#\#I}q = (\partial_t + I\partial_x - J\partial_y - K\partial_z)(t + \bar{r}) = 1 + I^2 - J^2 - K^2 = +2 \qquad (3.2.12)$$

Hence (3.2.10) gives,

$$\begin{aligned} dq^n &= (dq^{n-1})q + (q^{n-1})_0 dq + I(q^{n-1})_1 d^{\#\#I}q + J(q^{n-1})_2 d^{\#\#J}q \\ &\qquad + K(q^{n-1})_3 d^{\#\#K}q \\ &= (dq^{n-1})q - 2[(q^{n-1})_0 - I(q^{n-1})_1 - J(q^{n-1})_2 - K(q^{n-1})_3] \\ &= (dq^{n-1})q - 2(q^{n-1})^{\#} \end{aligned} \qquad (3.2.13)$$

This provides us with a recursion formula for finding dq^n from dq^{n-1}. Putting n=1 gives $dq = -2$, as it should (noting that $q^0 = 1$ and $dq^0 = 0$). Hence, we may now quickly and easily derive the derivatives of powers of q,

$$dq^2 = (dq)q - 2q^{\#} = -2q - 2q^{\#} = -4t$$

in agreement with (3.1.6), and hugely simpler than using Chain Rule No.1, i.e. (3.2.6,7). Similarly we find,

$$dq^3 = (dq^2)q - 2(q^2)^{\#} = -4tq - 2(t^2 - r^2 - 2t\bar{r}) = -6t^2 + 2r^2$$

in agreement with (3.1.9). Finally,

$$\begin{aligned} dq^4 &= (dq^3)q - 2(q^3)^{\#} \\ &= (-6t^2 + 2r^2)q - 2\big((t^2 - 3r^2)t - (3t^2 - r^2)\bar{r}\big) \\ &= -8t(t^2 - r^2) \end{aligned} \qquad (3.2.14)$$

It appears that dq^n is purely temporal (scalar) for any n.

We note that (3.2.13) holds also for negative n. Re-arranging and dividing by q gives,

$$dq^{n-1} = (dq^n).\frac{1}{q} + 2\frac{(q^n)^{\#}}{|q|^2} \qquad (3.2.15)$$

Putting $n = 0$ gives immediately,

$$d\left(\frac{1}{q}\right) = \frac{2}{|q|^2} \qquad (3.2.16)$$

a result which may be checked by direct evaluation (but is not so trivial, requiring a few lines of algebra to accomplish). Similar, with $n = -1$ we get,

$$d\left(\frac{1}{q^2}\right) = d\left(\frac{1}{q}\right).\frac{1}{q} + \frac{2}{|q|^2}.\frac{1}{q^{\#}} = \frac{2}{|q|^2}\left[\frac{1}{q} + \frac{1}{q^{\#}}\right] = \frac{4t}{|q|^4} \qquad (3.2.17)$$

which can again be checked by direct evaluation. Finally, for $n = -2$ and $n = -3$,

$$d\left(\frac{1}{q^3}\right) = d\left(\frac{1}{q^2}\right).\frac{1}{q} + \frac{2}{|q|^2}.\left(\frac{1}{q^2}\right)^{\#} = \frac{4t}{|q|^4}.\frac{1}{q} + \frac{2q^2}{|q|^6} = \frac{4tq^{\#} + 2q^2}{|q|^6} = \frac{6t^2 - 2r^2}{|q|^6} \qquad (3.2.18)$$

and, $$d\left(\frac{1}{q^4}\right) = \frac{8t(t^2 - r^2)}{|q|^8} \qquad (3.2.19)$$

By comparison of (3.2.16-19) with (3.1.2,6,9) and (3.2.14) it is evident that,

$$d\left(\frac{1}{q^n}\right) = -\frac{dq^n}{|q|^{2n}} \qquad (3.2.20)$$

Proof of this, and that both sides are scalar (temporal) for all n, is left as an exercise for the reader.

(3.2.13) may be written for an arbitrary function to provide another corollary of the second chain rule as,

$$d(Fq) = (dF)q - 2F^{\#} \quad (3.2.21)$$

The corresponding rule with q replaced by $q^{\#}$ is found by using the chain rule No.2 with the identity,

$$d^{\#\#I}q^{\#} = (\partial_t + I\partial_x - J\partial_y - K\partial_z)(t - \bar{r}) = 1 + 1 - 1 - 1 \equiv 0 \quad (3.2.22)$$

On the other hand,

$$dq^{\#} = (\partial_t + I\partial_x + J\partial_y + K\partial_z)(t - \bar{r}) = 1 + 1 + 1 + 1 = 4 \quad (2.3.23)$$

so that chain rule No.2 becomes, with the second function replaced by $q^{\#}$,

$$d(Fq^{\#}) = (dF)q^{\#} + 4f_0 \quad (3.2.24)$$

Other results for simple functions which follow either from direct evaluation or by using the above rules include,

$$d|q|^2 = 2q \quad \text{and hence} \quad d^{\#}|q|^2 = 2q^{\#} \quad (3.2.25)$$

$$d|q|^4 = 4q|q|^2 \quad \text{and hence} \quad d^{\#}|q|^4 = 4q^{\#}|q|^2 \quad (3.2.26)$$

$$d\left(\frac{1}{|q|^2}\right) = -\frac{2q}{|q|^4} \quad \text{and hence} \quad d^{\#}\left(\frac{1}{|q|^2}\right) = -\frac{2q^{\#}}{|q|^4} \quad (3.2.27)$$

$$d\left(\frac{1}{|q|^4}\right) = -\frac{4q}{|q|^6} \quad \text{and hence} \quad d^{\#}\left(\frac{1}{|q|^4}\right) = -\frac{4q^{\#}}{|q|^6} \quad (3.2.28)$$

$$d\left(\frac{1}{|q|^2 q}\right) \equiv d\left(\frac{q^{\#}}{|q|^4}\right) \equiv 0; \qquad d^{\#}\left(\frac{1}{|q|^2 q^{\#}}\right) \equiv d^{\#}\left(\frac{q}{|q|^4}\right) \equiv 0 \quad (3.2.29)$$

Note that the RHS of (3.2.27) are the same as the LHS of (3.2.29), modulo a constant factor. Hence we see that the reason why the quaternion-valued functions in (3.2.29) have identically zero gradients is that these functions are the (conjugate) gradients of $1/|q|^2 = 1/(t^2 + x^2 + y^2 + z^2)$ which satisfies Laplace's equation,

$$d^{\#}d\left(\frac{1}{|q|^2}\right) = dd^{\#}\left(\frac{1}{|q|^2}\right) = 0 \quad (3.2.30)$$

(3.2.9) establishes that $q^{\#}/|q|^4 = 1/|q|^2 q$ is a gradnull function. According to the theorem established above, there must therefore exist a function $f(x, y, z)$ such that it can be expressed as,

$$e^{-t\bar{\nabla}} f(x, y, z) \equiv \frac{1}{|q|^2 q} \quad (3.2.31)$$

If this is true then f must obviously be given by,

$$f(x, y, z) = \frac{-\bar{r}}{r^4} \quad (3.2.32)$$

(simply by considering $t = 0$). That (3.2.32) does yield $1/|q|^2 q$ when substituted into the LHS of (3.2.31) is not at all obvious. Evaluation of expressions of this type are considered below.

3.3 When is the Product of Two Gradnull Functions Gradnull?

In the case of complex functions, the product of two holomorphic, or analytic, functions is also holomorphic/analytic. The failure of the simple chain rule, (3.2.1), prevents this being the case for quaternionic gradnull functions. The chain rules that do hold, i.e. (3.2.4, 3.2.10), do not imply that $d(FG) = 0$ simply because $dF = 0$ and $dG = 0$. We therefore explore what additional conditions do permit this conclusion.

The most obvious is to require that the first function, F, is scalar (temporal) as well as gradnull. In that case we have $d\{F\} \equiv Fd$ and hence,

For scalar F: $$d(FG) = (dF)G + F(dG) \qquad (3.3.1)$$

and clearly then $dF = 0$ and $dG = 0$ implies $d(FG) = 0$, so the product function is also gradnull.

The same result holds if the second function, G, is scalar, since we can then just commute F and G,

For temporal G: $$d(FG) = d(GF) = (dG)F + G(dF) \qquad (3.3.2)$$

The same result follows from the proof, (3.2.11), of chain rule No.2, since, if G is scalar each of I, J, K may be pushed through to the right of the G-terms, changing all – signs to + in the process, and showing that $d(FG) = (dF)G + (dG)F = G(dF) + (dG)F$.

However, these results are quite useless because a purely scalar gradnull function must be a constant. This follows immediately from,

$$dF = df_0 = \partial_t f_0 + I\partial_x f_0 + J\partial_y f_0 + K\partial_z f_0 = 0$$

$$\Rightarrow \partial_t f_0 = 0,\ \partial_x f_0 = 0,\ \partial_y f_0 = 0,\ \partial_z f_0 = 0$$

(Note that the same is true for complex numbers: being analytic and real implies the function is a constant).

However, the restriction of one of the two functions to being scalar is unnecessarily strong. A weaker condition is sufficient to imply FG is analytic. Recall that all gradnull functions can be expressed as,

$$F = e^{-t\bar{\nabla}} f(x,y,z) \qquad \text{and} \qquad G = e^{-t\bar{\nabla}} g(x,y,z \qquad (3.3.3)$$

In general f and g are quaternionic (i.e. of the form $f = f_0 + If_1 + Jf_2 + Kf_3$, etc.). The action of the quaternion-grad operator on the product FG is thus,

$$d(FG) = (\partial_t + \bar{\nabla})\left[\left(e^{-t\bar{\nabla}} f(x,y,z)\right)\left(e^{-t\bar{\nabla}} g(x,y,z)\right)\right] \qquad (3.3.4)$$

We note that ∂_t commutes with f and g but not with the factors $e^{-t\bar{\nabla}}$, whereas $\bar{\nabla}$ commutes with the factors $e^{-t\bar{\nabla}}$ but not with f or g. Carrying out the time derivatives gives,

$$d(FG) = \left(-\bar{\nabla} e^{-t\bar{\nabla}} f(x,y,z)\right)\left(e^{-t\bar{\nabla}} g(x,y,z)\right) + \left(e^{-t\bar{\nabla}} f(x,y,z)\right)\left(-\bar{\nabla} e^{-t\bar{\nabla}} g(x,y,z)\right) + \bar{\nabla}\left[\left(e^{-t\bar{\nabla}} f(x,y,z)\right)\left(e^{-t\bar{\nabla}} g(x,y,z)\right)\right] \qquad (3.3.5)$$

We now impose the condition that f (rather than F) is scalar. In the last term of (3.3.5) this allows the simple chain rule to hold, i.e.,

$$\bar{\nabla}(fg) = (\bar{\nabla} f)g + f(\bar{\nabla} g) \qquad \text{for scalar } f \qquad (3.3.6)$$

Applying (3.3.6) in (3.3.5) shows the terms to cancel and we conclude,

For scalar f: $dF = 0$ and $dG = 0$ implies $d(FG) = 0$ (3.3.7)

Note that F in general will not be scalar merely because f is scalar, so (3.3.7) is a stronger result than (3.3.1).

What if g is temporal but not f? Then FG will ***not*** be gradnull in general. In fact it is easily shown that FG is gradnull only if either f is scalar or G is constant. This is because in (3.3.5) the first term cancels with that part of the third term in which the grad operates on f. The remaining part of the third term, pushing the operator through f but remembering that some factors anticommute, is,

$$\left[e^{-t\bar{\nabla}}(f_0 + If_1 - Jf_2 - Kf_3)Ie^{-t\bar{\nabla}}\partial_x G + e^{-t\bar{\nabla}}(f_0 - If_1 + Jf_2 - Kf_3)Je^{-t\bar{\nabla}}\partial_y G \right.$$
$$\left. + e^{-t\bar{\nabla}}(f_0 - If_1 - Jf_2 + Kf_3)Ke^{-t\bar{\nabla}}\partial_z G\right]$$

For $d(FG)$ to be identically zero this must cancel with the second term in (3.3.5) which is algebraically similar to the above but with all minus signs positive instead. A little thought shows that this can only be true if either, (a) f is purely scalar, or, (b) G is a constant.

This asymmetry in the significance of F and G arises because, by convention, a differential operator like d operates on items to its right – and because of the significance of which of F or G is on the left or right. If we invented an operator $\overleftarrow{d}$ which was like d but acted as a derivative to its left, then the roles of F and G would be reversed.

3.4 Evaluation of *F* from *f*

In this section we consider some examples of analytic functions found by specifying a function f and defining F via (3.3.3). Note that what this does is to create a function of all four variables, t, x, y, z, which is gradnull by construction (and which therefore is also a solution to the 4D Euclidean Laplace equation). Note that the reverse process is trivial. If we are given F and know that it is gradnull, then we know that such an f exists and its value is simply F evaluated at $t = 0$. We consider here some simple examples for f and derive the corresponding analytic F.

$\underline{f(x,y,z) = \bar{r}}$

Noting that $\bar{\nabla}\bar{r} = -\bar{\nabla}\cdot\bar{r} = -3$ it is clear that $\bar{\nabla}\bar{\nabla}\bar{r} \equiv 0$ and hence,

$$F = e^{-t\bar{\nabla}}\bar{r} = (1 - t\bar{\nabla})\bar{r} = \bar{r} - t(-3) = \bar{r} + 3t \quad (3.4.1)$$

Since F is gradnull by construction then $dF = 0$, and it is readily checked that this is true for $\bar{r} + 3t$.

$\underline{f(x,y,z) = r^2}$

Noting that $\bar{\nabla}r^2 = 2\bar{r}$ and hence that $\bar{\nabla}^2 r^2 = -6$ and $\bar{\nabla}^3 r^2 \equiv 0$ we find,

$$F = e^{-t\bar{\nabla}}r^2 = (1 - t\bar{\nabla} + (t^2/2)\bar{\nabla}^2)r^2 = r^2 - 2t\bar{r} - 3t^2 \quad (3.4.2)$$

and again it can be simply confirmed that $dF = 0$.

$\underline{f(x,y,z) = r^4}$

Noting that $\bar{\nabla}r^4 = 4r^2\bar{r}$ and hence that, (3.4.3)

$$\bar{\nabla}^2 r^4 = 4(\bar{\nabla}r^2)\bar{r} + 4r^2\bar{\nabla}\bar{r} = 4.2\bar{r}\bar{r} + 4r^2(-3) = -8r^2 - 12r^2 = -20r^2$$

and,

$$\bar{\nabla}^3 r^4 = -20\bar{\nabla} r^2 = -40\bar{r}, \quad \text{and} \quad \bar{\nabla}^4 r^4 = 120 \quad \text{and} \quad \bar{\nabla}^5 r^4 \equiv 0 \tag{3.4.4}$$

hence,

$$F = e^{-t\bar{\nabla}} r^4 = (1 - t\bar{\nabla} + (t^2/2)\bar{\nabla}^2 - (t^3/6)\bar{\nabla}^3 + (t^4/24)\bar{\nabla}^4) r^4$$
$$= r^4 - 4tr^2\bar{r} - 10t^2r^2 + \frac{20}{3}t^3\bar{r} + 5t^4 \tag{3.4.5}$$

which again can be checked to obey $dF = 0$.

$\underline{f(x,y,z) = r^2\bar{r}}$

In a similar manner we find,

$$F = e^{-t\bar{\nabla}} r^2\bar{r} = r^2\bar{r} + 5tr^2 - 5t^2\bar{r} - 5t^3 \tag{3.4.6}$$

and again $dF = 0$ can be checked directly.

It is clear that with $f = r^{2n}$ or $f = r^{2n}\bar{r}$ then F will be a finite polynomial. The same is true for odd powers of r after a little more work:-

$\underline{f(x,y,z) = r}$

We find, $\bar{\nabla} r = \frac{\bar{r}}{r}$ and $\bar{\nabla}^2 r = \frac{-2}{r}$ and $\bar{\nabla}^3 r = \frac{2\bar{r}}{r^3}$ and $\bar{\nabla}^4 r \equiv 0$. Hence we find,

$$F = e^{-t\bar{\nabla}} r = (1 - t\bar{\nabla} + (t^2/2)\bar{\nabla}^2 - (t^3/6)\bar{\nabla}^3 + (t^4/24)\bar{\nabla}^4) r$$
$$= r - \frac{t\bar{r}}{r} - \frac{t^2}{r} - \frac{t^3\bar{r}}{3r^3} \tag{3.4.7}$$

for which $dF = 0$ can be checked once again. It seems rather unexpected to find that $e^{-t\bar{\nabla}} r$ is such an expression – namely a cubic in t, rather than either linear or a infinite series, i.e., having found that $\bar{\nabla}^3 r$ is non-zero it is surprising to find that $\bar{\nabla}^4 r \equiv 0$. But the reason for this has an important physical meaning…

$\underline{f(x,y,z) = \bar{r}/r^3}$

We see from the above that $\bar{\nabla}(\bar{r}/r^3) \equiv 0$. This corresponds to the inverse-square field due to a point source solution of Laplace's equation, with corresponding potential $-1/r$, i.e.,

$$\nabla^2\left(\frac{1}{r}\right) = -\bar{\nabla}\bar{\nabla}\left(\frac{1}{r}\right) = \bar{\nabla}\left(\frac{\bar{r}}{r^3}\right) = 0 \tag{3.4.8}$$

Hence this f gives a static solution to the 4D Laplace equation,

$$F = e^{-t\bar{\nabla}}(\bar{r}/r^3) = f = \bar{r}/r^3 \tag{3.4.9}$$

$\underline{f(x,y,z) = r\bar{r}}$

We find that $\bar{\nabla}^5 r\bar{r} \equiv 0$ and,

$$F = e^{-t\bar{\nabla}} r\bar{r} = r\bar{r} + 4tr - 2t^2\frac{\bar{r}}{r} - \frac{4t^3}{3r} - \frac{t^4\bar{r}}{3r^3} \tag{3.4.10}$$

which again satisfies $dF = 0$.

$\underline{f(x,y,z) = 1/r}$

$$F = e^{-t\bar{\nabla}}\left(\frac{1}{r}\right) = 1/r + (-t)(-\bar{r}/r^3) = \frac{1}{r} + \frac{t\bar{r}}{r^3} \tag{3.4.11}$$

$\underline{f(x,y,z) = \bar{r}/r}$

$$F = e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r}\right) = \frac{\bar{r}}{r} - t\left(\frac{-2}{r}\right) + \frac{t^2}{2}\left(-2\frac{-\bar{r}}{r^3}\right) = \frac{\bar{r}}{r} + \frac{2t}{r} + \frac{t^2\bar{r}}{r^3} \tag{3.4.12}$$

<u>$f(x, y, z) = 1/r^2$</u>

This is the first of our examples to give rise to an infinite series (i.e., a transcendental function) in part of the result. It is evaluated with the aid of,

$$\bar{\nabla}\left(\frac{1}{r^2}\right) = \frac{-2\bar{r}}{r^4} \text{ and } \bar{\nabla}\left(\frac{\bar{r}}{r^4}\right) = \frac{1}{r^4} \text{ and } \bar{\nabla}\left(\frac{1}{r^4}\right) = \frac{-4\bar{r}}{r^6} \text{ and } \bar{\nabla}\left(\frac{\bar{r}}{r^6}\right) = \frac{3}{r^6}$$

$$\bar{\nabla}\left(\frac{1}{r^6}\right) = \frac{-6\bar{r}}{r^8} \text{ and } \bar{\nabla}\left(\frac{\bar{r}}{r^8}\right) = \frac{5}{r^8} \text{ and } \bar{\nabla}\left(\frac{1}{r^8}\right) = \frac{-8\bar{r}}{r^{10}} \tag{3.4.13}$$

giving,

$$e^{-t\bar{\nabla}}\left(\frac{1}{r^2}\right) = \left[\frac{1}{r^2} - \frac{t^2}{r^4} + \frac{t^4}{r^6} - \frac{t^6}{r^8} + \ldots..\right] + \frac{2\bar{r}}{r^3}\left[\frac{t}{r} - \frac{2t^3}{3r^3} + \frac{3t^5}{5r^5} - \frac{4t^7}{7r^7} + \ldots..\right] \tag{3.4.14}$$

The first series is simply $(t^2 + r^2)^{-1}$ whereas the derivative with respect to t of the second term is just $2\bar{r}(t^2 + r^2)^{-2}$. Integrating the latter using $t = r \cdot tan\theta$ gives,

$$e^{-t\bar{\nabla}}\left(\frac{1}{r^2}\right) = F = \frac{1}{t^2+r^2} + \frac{\bar{r}}{r^3}\left[\frac{rt}{t^2+r^2} + tan^{-1}\left(\frac{t}{r}\right)\right] \tag{3.4.15}$$

and once again it may be checked directly that $dF = 0$.

<u>$f(x, y, z) = \bar{r}/r^4$</u>

Recall that, because we have shown above that $1/|q|^2 q$ is gradnull, we should find, if all is well, that it can be written as $e^{-t\bar{\nabla}}(-\bar{r}/r^4)$. That this is indeed the case follows quickly from the precursors to the preceding example, i.e., from (3.4.13), thus,

$$e^{-t\bar{\nabla}}(\bar{r}/r^4) = \bar{r}\left[\frac{1}{r^4} - \frac{2t^2}{r^6} + \frac{3t^4}{r^8} - \frac{4t^6}{r^{10}} + \ldots..\right] - t\left[\frac{1}{r^4} - \frac{2t^2}{r^6} + \frac{3t^4}{r^8} - \frac{4t^6}{r^{10}} + \ldots..\right]$$

$$= \frac{(\bar{r}-t)}{r^4} \cdot \frac{1}{(1+t^2/r^2)^2} = \frac{\bar{r}-t}{(t^2+r^2)^2} = \frac{-q^\#}{|q|^4} = \frac{-1}{|q|^2 q} \tag{3.4.16}$$

<u>$f(x, y, z) = 1/r^3$</u>

This can be done using repeatedly,

$$\bar{\nabla}\bar{r} = -3 \quad \text{and} \quad \bar{\nabla}\left(\frac{1}{r^n}\right) = -n\frac{\bar{r}}{r^{n+2}} \tag{3.4.17}$$

which gives, (3.4.18)

$$e^{-t\bar{\nabla}}\left(\frac{1}{r^3}\right) = \left(\frac{1}{r^3} - 3\frac{t^2}{r^5} + 5\frac{t^4}{r^7} - 7\frac{t^6}{r^9} + \cdots\right) + \bar{r}\left(3\frac{t}{r^5} - 5\frac{t^3}{r^7} + 7\frac{t^5}{r^9} - 9\frac{t^7}{r^{11}} + \cdots\right)$$

which is easily shown to equal,

$$e^{-t\bar{\nabla}}\left(\frac{1}{r^3}\right) = \frac{r^2-t^2}{r(r^2+t^2)^2} + \frac{t(3r^2+t^2)}{r^3(r^2+t^2)^2}\bar{r} \tag{3.4.19}$$

4. Integration of Quaternionic Functions

4.1 Closed-Hypersurface Integrals over Quaternionic Functions (Introduction)

One of the most useful features of complex holomorphic functions is Cauchy's theorem. For a complex function which is holomorphic (analytic), containing no poles etc., on and within a closed contour Γ in the Argand plane, we have,

$$\oint_{\Gamma} f(z)dz = 0 \qquad (4.1.1)$$

This follows simply from the Cauchy-Riemann equations and Green's theorem. The latter is the identity,

$$\oint_{\Gamma} \left(f_x dx + f_y dy\right) \equiv \iint_{within\ \Gamma} dxdy \left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right] \qquad (4.1.2)$$

From which follows,

$$\oint_{\Gamma} f dz = \oint_{\Gamma} (f_r + if_i)(dx + idy) = \oint_{\Gamma} [(f_r dx - f_i dy) + i(f_i dx + f_r dy)] =$$

$$\iint_{within\ \Gamma} dxdy\{\left(-\partial_x f_i - \partial_y f_r\right) + i\left(\partial_x f_r - \partial_y f_i\right)\}$$

$$\equiv 0 \text{ by the Cauchy-Riemann equs. (2.1.1)}$$

It is reasonable to seek a generalisation of (4.1.1) for quaternionic gradnull functions. What is the equivalent of the closed contour Γ when dealing with the 4-dimensional quaternion space? The obvious choice is a closed hypersurface, δV_4, where V_4 represents a 4D region of quaternion space. Thus δV_4 is a (necessarily curved) 3-dimensional 'surface' which encloses the region V_4 of the 4-dimensional quaternion space. In what follows it must be remembered that this 4D space is Euclidean. It is not physical (Minkowski) spacetime with its indefinite metric.

In the following sub-sections we shall consider the following integral over the closed hypersurface δV_4,

$$\mathfrak{R} = \oiiint_{\delta V^4} d^3 S_q\, f(t,x,y,z) \qquad (4.1.3)$$

where f is any gradnull quaternionic function. (Note we have changed notation so that f is the gradnull function itself, not its generating function). $d^3 S_q$ is the quaternion-valued element of the 3D hypersurface δV_4. It has to be quaternion valued because it needs to be "vectorial" in the 4D Euclidian space. Just as a surface integral in 3D Euclidian space is a vector $d\bar{S}$ with components in all three coordinate directions, x, y, z, so $d^3 S_q$ has components in four coordinate directions according to whether the component is scalar (oriented temporally) or a 3-vector (oriented like $d\bar{S}$ with components in directions x, y, z). Details of how $d^3 S_q$ is calculated depend upon the choice of region, V_4, and hence its boundary, δV_4, and will be illustrated below.

Note that, because both $d^3 S_q$ and f are quaternion-valued, they will not commute, in general. Hence the order of these terms in (4.1.3) is crucial. Had we written the integral instead as,

$$\mathfrak{L} = \oiiint_{\delta V^4} f\,(t,x,y,z) d^3 S_q \qquad (4.1.4)$$

it would be a different integral. It will turn out that $\mathfrak{R}$, (4.1.3), is identically zero for any quaternionic function which is gradnull throughout V_4 and on its boundary. In contrast, that does not apply to $\mathfrak{L}$, (4.1.4), which will not be zero in general. However, we shall see how symmetry is restored later in §4.2.3.

4.2 The Quaternionic Grad Integral Theorem

In this section we shall consider the integral of (4.1.3) for any quaternionic function, f, not necessarily gradnull, and show that in all generality the hypersurface integral can be rewritten as an integral over the 4D volume of quaternion space of the quaternionic derivative of the function, df, i.e.,

$$\Re = \oiiint_{\delta V^4} d^3S_q\, f\,(t,x,y,z) = \int\iiint_{V^4} df(t,x,y,z)dtdxdydz \qquad (4.2.1)$$

This theorem is immediately reminiscent of the familiar divergence theorem. However, the quantities in (4.2.1) are quaternionic and hence have both scalar and vector parts. We shall see that (4.2.1) effectively amalgamates several standard vector integral theorems, of which the divergence theorem is only one.

From (4.2.1) it immediately follows that a gradnull function, i.e., $df \equiv 0$, must have $\Re = 0$ for any closed boundary δV_4, provided the function remains gradnull throughout its volume and on its surface.

The next subsection, §4.2.1, proves (4.2.1) for the case of a prismatic integration surface, δV_4. The section after, §4.2.2, shows that the theorem holds for an arbitrary closed surface, δV_4.

4.2.1 Proof of the Grad Theorem, (4.2.1), for a Prismatic Hypersurface

In this sub-section we prove (4.2.1) assuming an integration hypersurface, δV_4, which is prismatic – defined as follows,

- Consider a closed 2-surface δV_3 enclosing a 3D region V_3 of the spatial part of quaternion space, i.e., the 3-vector part;
- Part of the hypersurface δV_4 is made by extruding δV_3 along the 'time' axis from t_1 to t_2, creating a 4D 'prism' (often called a 'cylinder', though a non-round one) so that V_4 may be identified with $V_3 \otimes [t_1, t_2]$;
- The curved surface of the above 'prism' is turned into a closed hypersurface by adding 'caps' to its ends at times t_1 and t_2,. These 'caps' are simply the 3-volumes V_3 at times t_1 and t_2 respectively.

The quaternion-valued 3-surface element, d^3S_q, is defined in the obvious manner for each of these regions separately,

- For the end 'caps' the magnitude of d^3S_q is simply the usual spatial volume element $dxdydz$. Its normal therefore points in the temporal direction, and hence d^3S_q consists only of a temporal (scalar) part. At t_2 the outward normal is positive, whereas at t_1 the outward normal is negative. Hence $d^3S_q = d^3S_0 = dxdydz$ at t_2 but $d^3S_q = d^3S_0 = -dxdydz$ at t_1.
- For the curved surface of the hyper-cylinder, the normal to δV_4 is just the normal to δV_3. Hence if we write the normal 2D vector surface element of δV_3 as $d^2\bar{S}$ then we have simply $d^3S_q = d^2\bar{S}dt \equiv d^3\bar{S}$ (where the last is merely a definition). As usual, the vector notation can be reinterpreted as a quaternion, replacing the x, y, z unit vectors with I, J, K, and this is essential in the performance of the integral, (4.1.3). Note that d^3S_q has zero temporal part on the curved 'cylindrical' hypersurface.

Expanding the quaternion product, $d^3S_q\, f$, gives us,

$$\Re = \oiiint_{\delta V^4}\left[\left(f_0 d^3S_0 - \bar{f}\cdot d^3\bar{S}\right) + \left(f_0 d^3\bar{S} + \bar{f} d^3S_0 - \bar{f}\times d^3\bar{S}\right)\right] \qquad (4.2.2)$$

Note that the last term is preceded by a minus sign because, from (1.2.8), it is actually $+d^3\bar{S} \times \bar{f} \equiv -\bar{f} \times d^3\bar{S}$, i.e., the minus sign results from writing the differential d^3S_q first in the integrand of (4.2.1). Hence, the integral $\mathfrak{L}$, (4.1.4), in which the terms are in the reverse order, would equal the RHS of (4.2.2) except with $-\bar{f} \times d^3\bar{S}$ replaced by $+\bar{f} \times d^3\bar{S}$.

Consider firstly the scalar part of this integral, $\mathfrak{R}_0$. We may convert it to an integral over the 4-volume within V^4 as follows: The first term is only non-zero on the end caps (since d^3S_0 is zero on the curved part of the hypercylinder) and here d^3S_0 is simply the 3-volume element, thus,

$$\mathfrak{R}_0(\text{first term}) = \iiint_{V^3 \text{ at } t_2} f_0 dxdydz - \iiint_{V^3 \text{ at } t_1} f_0 dxdydz$$
$$= \iiint_{V^3} \int_{t_1}^{t_2} \frac{\partial f_0}{\partial t} dxdydzdt \qquad (4.2.3)$$

where the last form is simply the 4-volume integral over V^4. The second term in $\mathfrak{R}_0$ is non-zero only on the curved 'cylindrical' surface. It can also be converted to a 4-volume integral over V^4 by using the divergence theorem,

$$\mathfrak{R}_0(\text{second term}) = -\int_{t_1}^{t_2} dt \oiint_{\delta V^3} \bar{f} \cdot d^2\bar{S} = -\int_{t_1}^{t_2} dt \iiint_{V^3} \bar{\nabla} \cdot \bar{f}\, dxdydz \qquad (4.2.4)$$

Adding (4.2.3,4) gives,

$$\mathfrak{R}_0 = \int \iiint_{V^4} \left[\frac{\partial f_0}{\partial t} - \bar{\nabla} \cdot \bar{f}\right] dtdxdydz \qquad (4.2.5)$$

We note from (2.4.2) that the integrand in the above is simply the scalar component of df, and hence consistent with the theorem to be proved, (4.2.1). So far, so good. Now for the 3-vector part:-

We again convert the integrals in (4.2.2) into 4-volume integrals over V^4. The term in d^3S_0 is converted in the same way as (4.2.3), i.e.,

$$\oiiint_{\delta V^4} \bar{f} d^3S_0 \equiv \iiint_{V^3} [\bar{f}(t_2) - \bar{f}(t_1)]\, dxdydz \equiv \int \iiint_{V^4} \frac{\partial \bar{f}}{\partial t} dtdxdydz \qquad (4.2.6)$$

The terms in $d^3\bar{S}$ may be converted with the help of the following standard vector integral identities,

$$\oiint_{\delta V^3} A d^2\bar{S} \equiv \iiint_{V^3} \bar{\nabla} A\, \mathrm{d}^3V \quad \text{and} \quad \oiint_{\delta V^3} \bar{B} \times d^2\bar{S} \equiv -\iiint_{V^3} \bar{\nabla} \times \bar{B}\, \mathrm{d}^3V \qquad (4.2.7)$$

which give,

$$\oiiint_{\delta V^4} (f_0 d^3\bar{S} - \bar{f} \times d^3\bar{S}) = \int_{t_1}^{t_2} dt \oiint_{\delta V^3} (f_0 d^2\bar{S} - \bar{f} \times d^2\bar{S})$$
$$= \int_{t_1}^{t_2} dt \iiint_{V^3} (\bar{\nabla} f_0 + \bar{\nabla} \times \bar{f})\, dxdydz \qquad (4.2.8)$$

Adding (4.2.6,8) gives the 3-vector part of $\mathfrak{R}$ as,

$$\bar{\mathfrak{R}} = \int \iiint_{V^4} \left[\frac{\partial \bar{f}}{\partial t} + \bar{\nabla} f_0 + \bar{\nabla} \times \bar{f}\right] dtdxdydz \qquad (4.2.9)$$

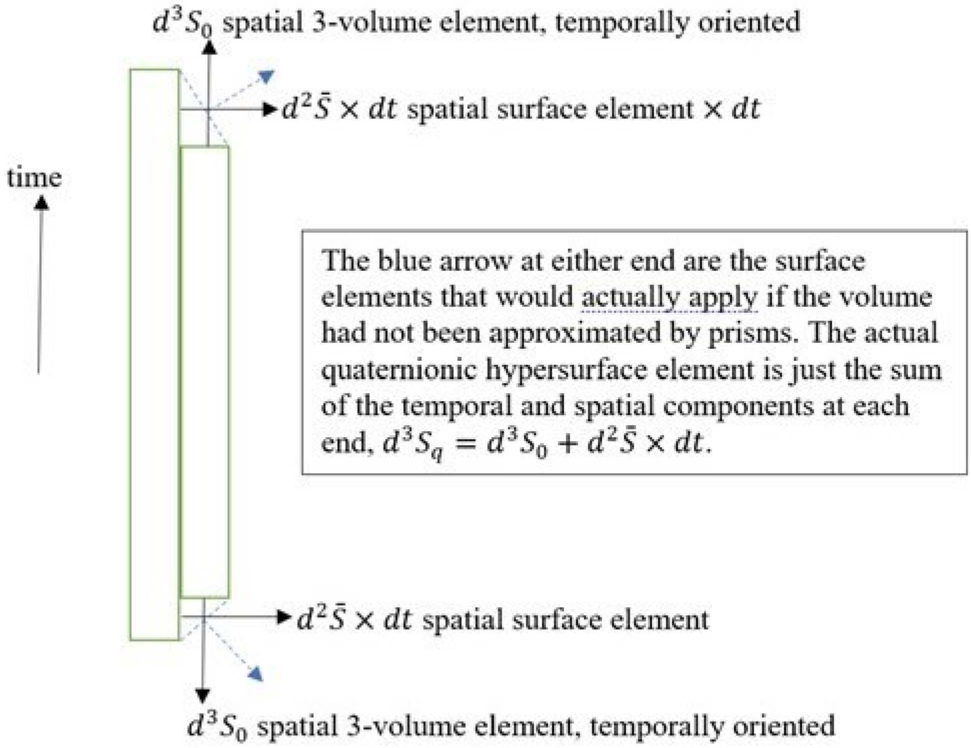
Comparison with (2.4.1) confirms that the integrand is indeed the 3-vector part of df. This completes the proof of (4.2.1) for a prismatic surface.

4.2.2 Proof of the Grad Theorem, (4.2.1), for an Arbitrary Closed Hypersurface

The proof proceeds as usual for these situations, namely that the arbitrary closed hypersurface is divided into a large number of narrow prismatic regions, oriented along the time direction (see Figure below). The theorem, (4.2.1), has been proved for each narrow prismatic region separately.

Summing over the $\oiiint_{\delta V^4} d^3S_q\, f$ due to each prism, their contributions cancel at their boundaries (as d^3S_q is equal and opposite there). The parts that do not cancel are, (i) the net contribution from the end caps, i.e., the difference between that at the two ends, and, (ii) the parts of the spatially oriented ('cylindrical') surfaces where there is overlap rather than cancellation. These two components are equivalent to the 4-vector of hypersurface element that would be obtained if we considered a smooth surface rather than one made up of "prismatic crystals".

Although the function f is, in principle, slightly different on the granular prismatic surface compared to the smooth surface, this difference makes only a second order difference to the integral, and in the limit is zero. The situation is illustrated below for one pair of adjacent prisms.



4.2.3 The Integral $\mathfrak{L}$, (4.1.4)

We note that the order of the quaternion product $d^3S_q\, f$ is what leads to the correct sign of the term $\bar{\nabla} \times \bar{f}$ in (4.2.9) and hence allows (4.2.1) to be deduced. In the variant integral, $\mathfrak{L}$, (4.1.4), the order is reversed, $f d^3S_q$, and this will lead to the same result for $\mathfrak{L}_0 = \mathfrak{R}_0$ and the same result for $\bar{\mathfrak{L}}$ as for $\bar{\mathfrak{R}}$ except for a minus sign in front of the term $\bar{\nabla} \times \bar{f}$ in (4.2.9). It follows immediately, therefore, that we can write the equivalent theorem for $\mathfrak{L}$, (4.1.4), by using a notation $\overleftarrow{d}$ which indicates that this differential operator acts on the function to its left. Hence, with the t, x, y, z dependence of f understood,

$$\mathfrak{L} = \oiiint_{\delta V^4} f\, d^3S_q = \int\iiint_{V^4} (f\overleftarrow{d})dtdxdydz \qquad (4.2.10)$$

where it is understood that, as far as the quaternion product is concerned, the $\overleftarrow{d}$ does indeed now stand to the right of f in $f\overleftarrow{d}$. In other words, the significance of the order of the quaternions in $d^3S_q\, f$ arises because of the convention that differential operators usually act on functions to their right.

4.2.4 A Closed Hypersurface Integral over a Gradnull Function is Zero

There is nothing left to prove. The Grad Integral Theorem, (4.2.1), immediately tells us that, if f is gradnull everywhere within V_4 and on its boundary, then,

$$\oiiint_{\delta V^4} d^3S_q\, f\,(t,x,y,z) = 0 \qquad (4.2.11)$$

5. A "Cauchy's Theorem" for Quaternionic Gradnull Functions

5.1 The Objective, What Fails and an Hypothesis

Cauchy's theorem for complex functions considers an integrand which contains a simple pole at a point z_0, constructed from a function f which is holomorphic everywhere within, and on, a closed contour Γ in the Argand plane, as follows,

$$\oint_\Gamma \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) \qquad (5.1.1)$$

where Γ is any closed contour which encloses z_0. It is the fact that (5.1.1) is contour independent that is its strength, and this results from the holomorphic nature of the whole integrand, $\frac{f(z)}{z-z_0}$, except at the isolated point z_0.

We now seek a generalisation of this for quaternionic gradnull functions with some sort of isolated point singularity. Because we wish the 3-surface integral to be finite and non-zero as the contour shrinks onto the singularity we may be tempted to consider the integral,

$$\oiiint_{\delta V^4} d^3S_q \frac{f(t,x,y,z)}{(q-q_0)^3} \qquad (5.1.2)$$

where f is a gradnull quaternion-valued function, and $q = t + Ix + Jy + Kz$, and q_0 is some particular position, $q_0 = t_0 + Ix_0 + Jy_0 + Kz_0$. The reasoning behind (5.1.2) is that a cubic denominator is necessary in order to cancel with the cubic hypersurface measure in the numerator, thus producing an integral which is neither divergent nor zero as the surface shrinks onto the point. (Recall that f will not, in general, be expressible as a function of q only).

However, it is clear that (5.1.2) does not meet our purpose because neither $1/(q-q_0)^3$ nor $\frac{f(t,x,y,z)}{(q-q_0)^3}$ in general are gradnull. So (5.1.2) will depend upon the integration surface. Moreover, as it happens, it is zero when evaluated on a hypersphere due to symmetry.

The latter problem – the vanishing of the integral on a hypersphere – can be cured by introducing angular dependence into the singular function. Thus we can consider,

$$\oiiint_{\delta V^4} d^3S_q \frac{1}{|q-q_0|^2(q-q_0)} f(t,x,y,z) = \oiiint_{\delta V^4} d^3S_q \frac{(q-q_0)^\#}{|q-q_0|^4} f(t,x,y,z) \qquad (5.1.3)$$

It turns out (as we show later) that this evaluates to $2\pi^2 f(0)$ on any hypersphere centred on the singular point, where $f(0) = f(t_0, x_0, y_0, z_0)$.

Unfortunately, as written, we have no reason to expect (5.1.3) to be independent of the integration surface because the integrand is not gradnull. The reason is that the generating function of $\frac{q^\#}{|q|^4}$ is $-\frac{\bar{r}}{r^4}$ which is not scalar and hence the product $\frac{(q-q_0)^\#}{|q-q_0|^4} f$ will not, in general, be gradnull even when f is gradnull.

The accompanying paper shows that a small tweak to (5.1.3) does render it independent of the integration region, giving us Fueter's Theorem,

$$\oiiint_{\delta V^4} \frac{1}{|q-q_0|^2(q-q_0)} d^3S_q f(t,x,y,z) = \oiiint_{\delta V^4} \frac{(q-q_0)^\#}{|q-q_0|^4} d^3S_q f(t,x,y,z) = 2\pi^2 f(0) \tag{5.1.4}$$

The paper also explains why this integral is surface independent, and how this independence arises in a distinct manner from that of my integral theorem.

However, we take a different approach and continue to look for a function, H, which is singular at a point and which has a real generating function so that Hf is gradnull when f is gradnull, thus providing an integral,

$$\oiiint_{\delta V^4} d^3S_q H(t-t_0, x-x_0, y-y_0, z-z_0) f(t,x,y,z) \propto f(0) \tag{5.1.5}$$

where $H = e^{-t\bar{\nabla}}(h)$ for a scalar generating function, h. What we have learnt from the preceding failed attempts is that, (i) H must be of inverse cubic order as $q \to q_0$, (ii) H must have some angular dependence to avoid becoming zero when integrated over a sphere, and (iii) the generating function, h, must be scalar so as to guarantee integration surface independence. These constraints lead us to considering the following hypothesis,

$$H(t,x,y,z) = e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) \tag{5.1.6}$$

We start by evaluating this function.

5.2 Evaluation of the Candidate Singular 'Projection' Function

In this section we evaluate $H = e^{-t\bar{\nabla}}(x/r^4)$ using the same methodology as in §3.4. $e^{-t\bar{\nabla}}(x/r^4)$ evaluates to a transcendental function, despite $e^{-t\bar{\nabla}}(\bar{r}/r^4) = \frac{-q^\#}{|q|^4}$ being so simple. The temptation to make the error of thinking the former is just the *I* component of the latter must be resisted. This example shows how very different they are.

With the aid of (3.4.17) and also,

$$\bar{\nabla}\left(\frac{\bar{r}}{r^n}\right) = \frac{n-3}{r^n} \tag{5.2.1a}$$

$$\bar{\nabla}\left(\frac{x}{r^n}\right) = \frac{I}{r^n} - n\frac{x\bar{r}}{r^{n+2}} \tag{5.2.1b}$$

It is readily shown that,

$$\bar{\nabla}\left(\frac{x}{r^4}\right) = \frac{I}{r^4} - 4\frac{x\bar{r}}{r^6} \tag{5.2.2}$$

$$\bar{\nabla}^2\left(\frac{x}{r^4}\right) = -4\frac{x}{r^6} \tag{5.2.3}$$

$$\bar{\nabla}^3\left(\frac{x}{r^4}\right) = -4\frac{I}{r^6} + 24\frac{x\bar{r}}{r^8} \tag{5.2.4}$$

$$\bar{\nabla}^4\left(\frac{x}{r^4}\right) = 6 \times 4 \times 3\frac{x}{r^8} \tag{5.2.5}$$

$$\bar{\nabla}^5\left(\frac{x}{r^4}\right) = 6 \times 4 \times 3\left(\frac{I}{r^8} - 8\frac{x\bar{r}}{r^{10}}\right) \tag{5.2.6}$$

$$\bar{\nabla}^6\left(\frac{x}{r^4}\right) = -8 \times 6 \times 5 \times 4 \times 3\frac{x}{r^{10}} \tag{5.2.7}$$

$$\bar{\nabla}^7\left(\frac{x}{r^4}\right) = -8 \times 6 \times 5 \times 4 \times 3\left(\frac{I}{r^{10}} - 10\frac{x\bar{r}}{r^{12}}\right) \tag{5.2.8}$$

which lead to, (5.2.9)

$$e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) = \frac{x}{r^4}\left[1 - 2\frac{t^2}{r^2} + 3\frac{t^4}{r^4} - 4\frac{t^6}{r^6} + \ldots\right] - \frac{It}{r^4}\left[1 - \frac{2}{3}\frac{t^2}{r^2} + \frac{3}{5}\frac{t^4}{r^4} - \frac{4}{7}\frac{t^6}{r^6} + \ldots\right]$$
$$+ \frac{4xt\bar{r}}{r^6}\left[1 - \frac{3}{3}\frac{t^2}{r^2} + \frac{6}{5}\frac{t^4}{r^4} - \frac{10}{7}\frac{t^6}{r^6} + \ldots\right]$$

The infinite sums in (5.2.9) are given by, with $\xi = t/r$,

$$[first] = \frac{1}{(1+\xi^2)^2} \tag{5.2.10}$$

$$[sec\,o\,nd] = \frac{1}{2}\left[\frac{1}{(1+\xi^2)} + \frac{1}{\xi}tan^{-1}\,\xi\right] \tag{5.2.11}$$

$$[third] = \left[\frac{5+3\xi^2}{8(1+\xi^2)^2} + \frac{3}{8\xi}tan^{-1}\,\xi\right] \tag{5.2.12}$$

Equs (5.2.11,12) are easily checked using,

$$tan^{-1}\,\xi = \xi - \frac{\xi^3}{3} + \frac{\xi^5}{5} - \frac{\xi^7}{7} + \ldots \tag{5.2.13}$$

which follows from,

$$\frac{d}{d\xi}(tan^{-1}\,\xi) = \frac{1}{1+\xi^2} \tag{5.2.14}$$

Hence we have,

$$H(t,x,y,z) \equiv e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) = \frac{x}{(t^2+r^2)^2} - I\frac{t}{2r^2}\left\{\frac{1}{t^2+r^2} + \frac{1}{rt}tan^{-1}\left(\frac{t}{r}\right)\right\} +$$
$$4xt\frac{\bar{r}}{r^4}\left\{\frac{5r^2+3t^2}{8(t^2+r^2)^2} + \frac{3}{8rt}tan^{-1}\left(\frac{t}{r}\right)\right\} \tag{5.2.15}$$

(5.2.15) at first seems surprising since the equivalent expression with the x on the LHS replaced by $\bar{r}$ is very simple, namely just $-q^{\#}/|q|^4$, see (3.4.16). The latter should be derivable from (5.2.15) by multiplying on the right by I and then adding the equivalent expressions in Jy and Kz, i.e.,

$$e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{Ix}{(t^2+r^2)^2} + \frac{t}{2r^2}\{2^{nd}\} + 4xt\frac{\bar{r}I}{r^4}\{3^{rd}\}$$
$$+ \frac{Jy}{(t^2+r^2)^2} + \frac{t}{2r^2}\{2^{nd}\} + 4yt\frac{\bar{r}J}{r^4}\{3^{rd}\}$$
$$+ \frac{Kz}{(t^2+r^2)^2} + \frac{t}{2r^2}\{2^{nd}\} + 4zt\frac{\bar{r}K}{r^4}\{3^{rd}\} \tag{5.2.16}$$

where the {..} are the corresponding bracketed terms in (5.2.15). Anticommutation ensures that,

$$x\bar{r}I + y\bar{r}J + z\bar{r}K = -r^2 \tag{5.2.17}$$

Hence (5.2.16) becomes,

$$e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{\bar{r}}{(t^2+r^2)^2} + \frac{t}{2r^2}[3\{2^{nd}\} - 8\{3^{rd}\}] \tag{5.2.18}$$

Using the expressions for the { } from (5.2.15) gives,

$$[3\{2^{nd}\} - 8\{3^{rd}\}] = -\frac{t}{(t^2+r^2)^2} \tag{5.2.19}$$

the awkward tan^{-1} terms cancelling. Hence we have,

$$e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{\bar{r}}{(t^2+r^2)^2} - \frac{t}{(t^2+r^2)^2} = \frac{-q^\#}{|q|^4} \tag{5.2.20}$$

in agreement with (3.4.16). Note the crucial difference between observing, on the one hand, that $e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{-q^\#}{|q|^4}$ can be derived from (5.2.15) by multiplying on the right by I and then adding the equivalent expressions in Jy and Kz – as we have just shown does work – and, on the other hand, mistakenly thinking that $e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right)$ could be obtained from $e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right)$ simply as the I component. In the former case, the expressions (5.2.15) contain terms, e.g., the tan^{-1} terms, which cancel in the summation – and which could never, therefore, arise from $e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right)$.

Having evaluated $e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right)$ we now need to calculate the integral,

$$\oiiint_{\delta V^4} d^3S_q e^{-(t-t_0)\bar{\nabla}}\left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right) f(t,x,y,z) \tag{5.2.21}$$

to see if it is indeed just a fixed constant times $f(0)$. As the integrand is gradnull, by construction, we already know that this integral is independent of its surface of integration, provided it is closed and contains q_0. As the surface shrinks onto q_0, therefore, it can only depend upon $f(0)$. But we could be disappointed and find that it is identically zero, as we found for (5.1.2).

5.3 Evaluating the Candidate 'Cauchy' Integral

We firstly calculate (5.2.21) on a hypersphere centred on q_0, and then we calculate it again using a prismatic surface. In both cases we shall need to shrink the surface onto the point q_0 so that f can be replaced by the constant $f(0)$. The two answers should, of course, be the same as the integral is expected to be surface-independent.

5.3.1 Evaluation on a Hypersphere

Before launching into the integral itself, we need to address the geometry of a hypersphere, and determine what the quaternionic surface element, d^3S_q, is on such a hypersphere.

To facilitate this we use 4D polar coordinates, with α the angle that the "4D vector" (i.e., the quaternion) makes with the time axis, and the other angles as per the usual 3D spherical polars. On a hypersphere of radius ρ we may therefore write the 'position quaternion', q, in terms of a unit 'radial' quaternion, u, as,

$$q = \rho u \qquad u = e^{\alpha\hat{u}} = \cos\alpha + \hat{u}\sin\alpha \tag{5.3.1}$$

where $\hat{u}$ is the unit radial vector in 3D space. In 3D polars the latter may be written,

$$\hat{u} = I\sin\theta\cos\varphi + J\sin\theta\sin\varphi + K\cos\theta \tag{5.3.2}$$

The position vector in 3D space is thus,

$$\bar{r} = r\hat{u} = r(I\sin\theta\cos\varphi + J\sin\theta\sin\varphi + K\cos\theta) \tag{5.3.3}$$

where r is the projection of the 4D 'length', ρ, onto 3D space, i.e.,

$$r = \rho\sin\alpha \qquad \text{and similarly,} \qquad t = \rho\cos\alpha \tag{5.3.4}$$

The x, y, z coordinates written explicitly are,

$$x = \rho\sin\alpha\sin\theta\cos\varphi\,, \quad y = \rho\sin\alpha\sin\theta\sin\varphi\,, \quad z = \rho\sin\alpha\cos\theta \tag{5.3.5}$$

We now derive an expression for the quaternion hypersurface element, d^3S_q, on the hypersphere. We note that its magnitude is just that of the ordinary 3D hypersurface element on the surface of a 4D sphere, i.e., d^3S, and its 'direction' is the 'radial' direction in 4D quaternion space, i.e. as given by the unit radial quaternion u. Hence we have $d^3S_q = ud^3S$. Expressions for u have been given explicitly above. It remains only to find d^3S. This is done by considering the small element of volume formed by small increases in the coordinates $\rho, \alpha, \theta, \varphi$, i.e., $\delta\rho, \delta\alpha, \delta\theta, \delta\varphi$. The increases in the t, x, y, z coordinates due to the increases in the 4D polar coordinates are found by differentiating (5.3.4,5) as follows, (in rather unconventional order),

$$ds_\alpha = \begin{pmatrix} dt \\ dz \\ dy \\ dx \end{pmatrix} = \rho \begin{pmatrix} -\sin\alpha \\ \cos\alpha\cos\theta \\ \cos\alpha\sin\theta\sin\varphi \\ \cos\alpha\sin\theta\cos\varphi \end{pmatrix} d\alpha \quad \text{for constant } \rho, \theta, \phi \tag{5.3.6}$$

$$ds_\theta = \begin{pmatrix} dt \\ dz \\ dy \\ dx \end{pmatrix} = \rho \begin{pmatrix} 0 \\ -\sin\alpha\sin\theta \\ \sin\alpha\cos\theta\sin\varphi \\ \sin\alpha\cos\theta\cos\varphi \end{pmatrix} d\theta \quad \text{for constant } \rho, \alpha, \phi \tag{5.3.7}$$

$$ds_\varphi = \begin{pmatrix} dt \\ dz \\ dy \\ dx \end{pmatrix} = \rho \begin{pmatrix} 0 \\ 0 \\ \sin\alpha\sin\theta\cos\varphi \\ -\sin\alpha\sin\theta\sin\varphi \end{pmatrix} d\varphi \quad \text{for constant } \rho, \alpha, \theta \tag{5.3.8}$$

$$ds_\rho = \begin{pmatrix} dt \\ dz \\ dy \\ dx \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\theta \\ \sin\alpha\sin\theta\sin\varphi \\ \sin\alpha\sin\theta\cos\varphi \end{pmatrix} d\rho \quad \text{for constant } \alpha, \theta\,, \phi \tag{5.3.9}$$

The four 4-vectors $ds_\alpha, ds_\theta, ds_\varphi, ds_\rho$ are seen to be mutually orthogonal. Consequently, the element of volume formed by them is just the product of their magnitudes. Their magnitudes are found from the above expressions to be,

$$ds_\rho = d\rho, \quad \mathrm{ds}_\alpha = \rho d\alpha, \quad \mathrm{ds}_\theta = \rho\sin\alpha\,.\,d\theta, \quad \mathrm{ds}_\varphi = \rho\sin\alpha\sin\theta\,.\,d\varphi \tag{5.3.10}$$

Hence the 4D volume element is simply,

$$d^4V = \rho^3\sin^2\alpha\,.\sin\theta\,.\,d\rho d\alpha d\theta d\varphi \tag{5.3.11}$$

and the scalar magnitude of the 3D hypersurface area element on the hypersphere is $d^3S = d^4V/d\rho$, i.e.,

$$d^3S = \rho^3 \sin^2\alpha . \sin\theta . d\alpha d\theta d\varphi \tag{5.3.12}$$

Hence we have the quaternion ("4-vector") of the element of the hypersphere

$$d^3S_q = u\rho^3 \sin^2\alpha . \sin\theta . d\alpha d\theta d\varphi \tag{5.3.13}$$

where the quaternion unit 'radial' 4-vector, u, is given explicitly by (5.3.1,2).

We are now in a position to calculate the integral (5.2.21) on a small hypersphere. Substituting (5.3.13) and (5.3.1) into (5.2.21) and considering only the limit of small ρ, we need to evaluate, (5.3.14)

$$\mathfrak{R} = \oiiint_{sphere\ \rho\to 0} \rho^3 \sin^2\alpha \sin\theta\, d\alpha d\theta d\varphi . (\cos\alpha + \hat{u}\sin\alpha) H(q - q_0) f(0)$$

where $H(q - q_0)$ is the function given explicitly by (5.2.15). We have taken liberties with the notation here in that $H(q - q_0)$ is ***not*** a function of a quaternion but separately of its four components. More properly it should be written,

$$H(t - t_0, x - x_0, y - y_0, z - z_0)$$

In fact the form of (5.2.15) is such that it can be written also as,

$$H(t - t_0, x - x_0, |\bar{r} - \bar{r}_0|)$$

Note that in order for the singularity to be an isolated point in the 4D quaternion space it is necessary that the singularity resides at one 'instant', $t = t_0$, as well as at one spatial point, $\bar{r} = \bar{r}_0$.

In the limit $\rho \to 0$ the function is just a constant, $f(0)$, which we will omit from further evaluations of (5.3.14) until the end. We evaluate it by considering the four combinations of scalar and vector parts arising from the product uH.

<u>(S x S)</u>: The contribution to (5.3.14) of the product of the two scalar parts is,

$$SS = \int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \cos\alpha\, \rho^3 \sin^2\alpha\, d\alpha \sin\theta\, d\theta d\varphi \frac{\rho \sin\alpha \sin\theta \cos\varphi}{\rho^4} \tag{5.3.15}$$

where we have used (5.3.5) for x. The ρ dependence cancels, as it should. The integral is clearly zero, both because of the φ integration, and also the α integration (the integrand being odd about the centre of the α-range). Hence,

$$SS \equiv 0 \tag{5.3.16}$$

<u>(S x V):</u> The contribution to (5.3.14) of the product of the scalar part of u and the vector part of H is evaluated by taking each of the two vector parts of H given in (5.2.15) in turn… (5.3.17)

$$SV1 = \int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \cos\alpha\, \rho^3 \sin^2\alpha\, d\alpha \sin\theta\, d\theta d\varphi \frac{-I}{2\rho^3}\frac{\cos\alpha}{\sin^2\alpha}\left(1 + \frac{\pi/2-\alpha}{\sin\alpha\cos\alpha}\right)$$

where we have used,

for $0 \le \alpha \le \pi$: $\tan^{-1}(t/r) = \tan^{-1}(\cot\alpha) = \pi/2 - \alpha$ (5.3.18)

Carrying out the θ and φ integrals, (5.3.17) becomes,

$$SV1 = -2\pi I \int_0^{\pi}\left[\cos^2\alpha + \frac{(\pi/2-\alpha)\cos\alpha}{\sin\alpha}\right] d\alpha = -I\pi^2 - 2\pi I \mathfrak{I}_{inf} \tag{5.3.19}$$

Where, $$\Im_{inf} = \int_0^{\pi} \left[\frac{(\pi/2-\alpha)\cos\alpha}{\sin\alpha}\right] \quad (5.3.20)$$

In fact, $\Im_{\text{inf}}$ is a divergent integral, but it will be seen that it cancels with the next term. The second part of the S x V term is, (5.3.21)

$$SV2 = \int_{\alpha=0}^{\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \cos\alpha\, \rho^3 \sin^2\alpha\, d\alpha \sin\theta\, d\theta d\varphi \frac{4\rho \sin\alpha \sin\theta \cos\varphi\, \rho\cos\alpha}{\rho^4 \sin^4\alpha} \times \left\{\frac{3+2\sin^2\alpha}{8\rho^2} + \frac{3(\pi/2-\alpha)}{8\rho^2 \sin\alpha\cos\alpha}\right\} \rho\sin\alpha\,(I\sin\theta\cos\varphi + J\sin\theta\sin\varphi + K\cos\theta)$$

The J and K terms are zero by virtue of the φ-integral. Carrying out the θ and φ integrals in the I term gives,

$$\begin{aligned} SV2 &= \frac{4\pi I}{3}\int_{\alpha=0}^{\pi} d\alpha \left\{\frac{3}{2}\cos^2\alpha + \cos^2\alpha\sin^2\alpha + \frac{3}{2}\frac{(\pi/2-\alpha)\cos\alpha}{\sin\alpha}\right\} \\ &= 2\pi I \int_{\alpha=0}^{\pi} d\alpha\left\{\cos^2\alpha + \frac{2}{3}\cos^2\alpha\,(1-\cos^2\alpha) + \frac{(\pi/2-\alpha)\cos\alpha}{\sin\alpha}\right\} \\ &= 2\pi I \int_{\alpha=0}^{\pi} d\alpha\left\{\frac{5}{3}\cos^2\alpha - \frac{2}{3}\cos^4\alpha + \frac{(\pi/2-\alpha)\cos\alpha}{\sin\alpha}\right\} \\ &= 2\pi I\Im_{\text{inf}} + \frac{2}{3}\pi I\left(5\frac{\pi}{2} - 2\frac{3\pi}{8}\right) \\ &= 2\pi I\Im_{\text{inf}} + \frac{7}{6}I\pi^2 \end{aligned} \quad (5.3.22)$$

where $\Im_{\text{inf}}$ is given by (5.3.20), and is divergent. However, adding (5.3.19) and (5.3.22) these divergent terms cancel and we get,

$$SV_{total} = SV1 + SV2 = I\frac{\pi^2}{6} \quad (5.3.23)$$

<u>(V x S):</u> The contribution to (5.3.14) of the product of the vector part of u and the scalar part of H is,

$$QT = \int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \rho^3 \sin^2\alpha\, d\alpha \sin\theta\, d\theta d\varphi \sin\alpha\,(I\sin\theta\cos\varphi + J\sin\theta\sin\varphi + K\cos\theta) \times \rho\sin\alpha\sin\theta\cos\varphi/\rho^4 \quad (5.3.24)$$

The J and K terms are zero by virtue of the φ integration, leaving,

$$\begin{aligned} QT &= I\int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \sin^4\alpha\, d\alpha \sin^3\theta\, d\theta \cos^2\varphi d\varphi \\ &= I\left(\frac{3\pi}{8}\right)\left(\frac{4}{3}\right)\left(\frac{1}{2}.2\pi\right) = I\frac{\pi^2}{2} \end{aligned} \quad (5.3.25)$$

<u>(V x V):</u> The contribution to (5.3.14) of the product of the vector part of u and the vector part of H is evaluated taking each of the two vector parts of H given in (5.2.15) in turn. The first term gives,

$$VV1 = \int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \rho^3\sin^2\alpha\, d\alpha\sin\theta\, d\theta d\varphi \sin\alpha\,(I\sin\theta\cos\varphi + J\sin\theta\sin\varphi + K\cos\theta) \times \left(\frac{-I}{2\rho^3}\frac{\cos\alpha}{\sin^2\alpha}\right)\left[1 + \frac{(\pi/2-\alpha)}{\sin\alpha\cos\alpha}\right] \quad (5.3.26)$$

The I and J terms are zero due to the φ integral, whereas the K term is zero due to the θ integral. Hence VV1 = 0.

The second vector term in H gives, (5.3.27)

$$VV2 = \int_{\alpha=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} sin^3\,\alpha\, d\alpha\, sin\,\theta\, d\theta d\varphi\, (I\, sin\,\theta\, cos\,\varphi + J\, sin\,\theta\, sin\,\varphi + K\, cos\,\theta) \times 4\, sin\,\alpha\, sin\,\theta\, cos\,\varphi \frac{cos\,\alpha}{sin^4\,\alpha}\left\{\frac{3 + 2\, sin^2\,\alpha}{8} + \frac{3(\pi/2 - \alpha)}{8\, sin\,\alpha\, cos\,\alpha}\right\} \times sin\,\alpha\,(i\, sin\,\theta\, cos\,\varphi + j\, sin\,\theta\, sin\,\varphi + k\, cos\,\theta)$$

The cross terms in IJ, JK, KI cancel due to anticommutation. The squared terms in the integrand I^2, J^2, K^2 are all -1 and the resulting trig expression is identically -1. Hence the only φ dependence in the integral we are left with is one factor of $cos\,\varphi$. Hence $VV2 = 0$ also, and,

$$VV_{total} = 0 \tag{5.3.28}$$

Summing the four contributions, (5.3.16,23,25,28) shows the integral (5.3.14) on an infinitesimal sphere to evaluate to,

$$\oiiint_{\text{hypersphere } \rho\to 0} d^3S_q e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) f(t, x, y, z) = I\pi^2\left(\frac{1}{6} + \frac{1}{2}\right) f(0) = \frac{2}{3} I\pi^2 f(0) \tag{5.3.29}$$

More importantly, for any gradnull function, $f(t, x, y, z)$, the integral over any closed hypersurface enclosing the origin will have the same result. In (5.3.29) I have written the singular point as the origin, but this can be moved to any point with the same result.

Finally, then, we have our generalisation of Cauchy's theorem applicable for an arbitrary gradnull function $f(t, x, y, z)$,

$$\oiiint_{\delta V^4} d^3S_q \left[e^{-(t-t_0)\bar{\nabla}}\left(\frac{x - x_0}{|\bar{r} - \bar{r}_0|^4}\right)\right] f(t, x, y, z) \equiv \frac{2}{3} I\pi^2 f(t_0, x_0, y_0, z_0) \tag{5.3.30}$$

where δV^4 is any closed hypersurface containing the point $q_0 = t_0 + \bar{r}_0$. If δV^4 does not contain q_0 then the integral is identically zero. By symmetry we can immediately write down also,

$$\oiiint_{\delta V^4} d^3S_q \left[e^{-(t-t_0)\bar{\nabla}}\left(\frac{y - y_0}{|\bar{r} - \bar{r}_0|^4}\right)\right] f(t, x, y, z) \equiv \frac{2}{3} J\pi^2 f(t_0, x_0, y_0, z_0) \tag{5.3.31}$$

$$\oiiint_{\delta V^4} d^3S_q \left[e^{-(t-t_0)\bar{\nabla}}\left(\frac{z - z_0}{|\bar{r} - \bar{r}_0|^4}\right)\right] f(t, x, y, z) \equiv \frac{2}{3} K\pi^2 f(t_0, x_0, y_0, z_0) \tag{5.3.32}$$

5.3.2 Evaluation on a Long, Thin Cylinder

Here we take the integration surface to be $S^2 \otimes [-t_0, +t_0]$, plus the associated end caps, in the limit $t_0 \to \infty$ and also $\rho \to 0$, where ρ is the radius of the 2-sphere. There is a subtlety here that should be exposed. If we were to include the function f is the evaluation we would be obliged to end up with an integral over the time coordinate between infinite limits involving $f(t, \bar{r} = 0)$, which would defy simplification. However, the reasoning goes like this: by considering integration over an infinitesimal 3-sphere we already know that the result in that case depends only on $f(t = 0, \bar{r} = 0)$. So the same is true for any closed integration surface including the origin because $e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) f$ is gradnull everywhere else. And the function $e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right)$ is also gradnull everywhere except the origin, so we can equate its integral over the 3-sphere to that over a long thin cylinder – and so that's all we need. We do not have to include f.

End Caps

The end caps are the ordinary 3D spatial volume integral within a sphere of radius ρ at $t = t_0$ minus that at $t = -t_0$. Recall we are integrating the function,

$$H(t,x,y,z) \equiv e^{-t\bar{\nabla}}\left(\frac{x}{r^4}\right) = \frac{x}{(t^2+r^2)^2} - I\frac{t}{2r^2}\left\{\frac{1}{t^2+r^2} + \frac{1}{rt}tan^{-1}\left(\frac{t}{r}\right)\right\} + 4xt\frac{\bar{r}}{r^4}\left\{\frac{5r^2+3t^2}{8(t^2+r^2)^2} + \frac{3}{8rt}tan^{-1}\left(\frac{t}{r}\right)\right\} \quad (5.2.15)$$

The first term (the scalar) is zero on the end caps as $t_0 \to \infty$. The same is true for the first terms within each of the two {...}. For $t_0 \to \pm\infty$ the $tan^{-1}\left(\frac{t}{r}\right)$ terms are $\pm\pi/2$. As the integrand is equal and opposite on the two end caps the two ends do *not* cancel. That leaves us with the requirement to integrate spatially over the function,

$$-I\frac{\pi}{4r^3} + \frac{3\pi x\bar{r}}{4r^5}$$

The first term integrates to $-I\pi^2\int_0^\rho \frac{dr}{r}$ which is divergent. However the second term integrates as follows,

$$\frac{3\pi}{4}\int \frac{1}{r^5} rsin\theta cos\varphi . r(Isin\vartheta cos\varphi + Jsin\theta sin\varphi + Kcos\theta)r^2 dr. sin\theta d\theta d\varphi$$

The φ integral kills the J and K terms and we are left with,

$$\frac{3\pi}{4}I\int_0^\rho \frac{dr}{r}\int_0^\pi sin^3\theta d\theta \int_0^{2\pi} cos^2\varphi d\varphi = \frac{3\pi}{4}I\cdot\frac{4}{3}\cdot\pi\int_0^\rho \frac{dr}{r} = I\pi^2\int_0^\rho \frac{dr}{r}$$

which cancels with the first term, above. So the end caps integrate to zero.

Cylindrical Surface

Here we have,

$$d^3S_q = dtd^2S_q = dt\hat{r}r^2 sin\theta d\theta d\varphi$$
$$= (Isin\vartheta cos\varphi + Jsin\theta sin\varphi + Kcos\theta)dtr^2 sin\theta d\theta d\varphi$$

Hence, considering firstly the scalar term in the integrand, (5.2.15), we require,

$$\int (Isin\vartheta cos\varphi + Jsin\theta sin\varphi + Kcos\theta)\frac{rsin\theta cos\varphi}{(t^2+r^2)^2}dtr^2 sin\theta d\theta d\varphi$$

The φ integral kills the J and K terms and we are left with,

$$I\int_0^\pi sin^3\theta d\theta \int_0^{2\pi} cos^2\varphi d\varphi \int_{-\infty}^{+\infty}\frac{r^3}{(t^2+r^2)^2}dt = \frac{4\pi}{3}I\int_{-\infty}^{+\infty}\frac{r^3}{(t^2+r^2)^2}dt = \frac{2}{3}\pi^2 I$$

noting that $\int_{-\infty}^{+\infty}\frac{r^3}{(t^2+r^2)^2}dt = \frac{\pi}{2}$. This is, in fact, the final answer – and agrees with (5.3.30) – because the two vector parts of (5.2.15) integrate to zero.

For the first of the vector terms this is immediately clear because it contains no angular dependence, and so inevitably integrates to zero over the vectorial integration measure $d^2S_q \propto \hat{r}$.

For the second vector term, its product with d^2S_q involves $\hat{r}^2 = -1$, which eliminates its angular dependence. The remaining angular-dependent terms are $x.sin\theta d\theta d\varphi = rsin^2\theta cos\varphi d\theta d\varphi$ which is killed by the φ integral.

Hence, integration over the long, thin cylinder reproduces (5.3.30), as expected.

QED.

5.3.3 Derivation of Fueter's Theorem

See the accompanying paper on this site.

6. Biquaternions: Basic Formulation

The general quaternion is $w + xI + yJ + zK$ where w, x, y, z are real. The general biquaternion is $q = w + xI + yJ + zK$ where w, x, y, z may be complex.

The hash notation denotes the quaternion-conjugate, as before, such that,

$$q^{\#} = w - xI - yJ - zK \tag{6.1}$$

noting that this leaves the (generally complex) coordinates, w, x, y, z, unchanged. In contrast, the complex-conjugate is, as normal,

$$q^{*} = w^{*} + x^{*}I + y^{*}J + z^{*}K \tag{6.2}$$

Note that, for any two biquaternions, $(pq)^{\#} = q^{\#}p^{\#}$, which corresponds to the similar expression for the Hermitian conjugate of complex matrices. In contrast, $(pq)^{*} = p^{*}q^{*}$.

When dealing with biquaternions, the terms "real" and "imaginary" become ambiguous. I use these terms to relate solely to the complex context. For the quaternionic part I refer to w as the "scalar" or "temporal" part (whether it is real or complex), and the $xI + yJ + zK$ part as the "vector" or "spatial" part, whether the coefficients are real or complex. Hence, the term "the real part of q" would mean $\mathcal{R}(w) + \mathcal{R}(x)I + \mathcal{R}(y)J + \mathcal{R}(z)K$. In contrast the scalar part is denoted $\mathbb{S}(q) = w$ and the vector part is denoted $\mathbb{V}(q) = xI + yJ + zK$, both of which may, in general, be complex.

Whilst the quaternions form a division ring, as every non-zero quaternion has an inverse, the biquaternions do not form a division ring as not every non-zero element has an inverse (for example, $1 + iI$ has no inverse). For non-zero quaternions there is a positive definitive norm whose square is defined by $N(q) = qq^{\#} = w^2 + x^2 + y^2 + z^2$ so that $q^{-1} = q^{\#}/N(q)$. In contrast, for biquaternions, $qq^{\#}$ will not generally be real, or positive definite even when it is real (though it is scalar), whilst $qq^{*\#}$ is not generally scalar (though it is real). The lack of an inverse for the general biquaternion is essential for its utility in relativity since it relates directly to the Minkowski metric not being positive definite.

6.1 Minkowski Spacetime and Lorentz Transformations

Spacetime points (events) are denoted by special biquaternions, referred to as Hermitian biquaternions, defined as having a real scalar part and a purely imaginary vector part. Thus, if t is a real time coordinate and $\bar{r}$ is a real 3-vector, which in quaternion form is written $\bar{r} = xI + yJ + zK$, where x, y, z are real, then the event is represented by $q = t + i\bar{r}$.

Hence, Hermitian biquaternions are defined such that they equal their full conjugate, $q^{*\#} = q$ (equivalently $q^{*} = q^{\#}$). The squared-norm of an Hermitian biquaternion is thus, (6.1.1)

Hermitian biquaternions: $N(q) = qq^{\#} = (t + i\bar{r})(t - i\bar{r}) = t^2 - [x^2 + y^2 + z^2]$

where, despite appearances, the minus sign characteristic of the Minkowski metric, occurs because of $I^2 = J^2 = K^2 = -1$. This immediately links to the Lorentz transformation which can be defined as the most general transformation which preserves the norm of an Hermitian biquaternion.

It follows that the transformation of an arbitrary Hermitian biquaternion, p, defined by,

$$p \to p' = qpq^{*\#} \tag{6.1.2}$$

where q is any biquaternion with unit norm (i.e., with $q^{\#}q = qq^{\#} = 1$) is a Lorentz transformation. This is because,

$$p'p'^{\#} = qpq^{*\#}(qpq^{*\#})^{\#} = qpq^{*\#}q^{*}p^{\#}q^{\#} = qpp^{\#}q^{\#} = pp^{\#}qq^{\#} = pp^{\#}$$

thus showing that the norm (which is the Lorentz scalar product for an Hermitian biquaternion) is invariant, as required. Note we have used the fact that $q^{\#}q = 1$ implies $q^{*\#}q^{*} = 1$ and that $pp^{\#}$ is a scalar for an Hermitian biquaternion.

If we write $q = u + iv$, where u and v are (real) quaternions, then the requirement for q to represent a Lorentz transformation, i.e., $qq^{\#} = 1$, becomes $uu^{\#} - vv^{\#} = 1$ and $uv^{\#} + vu^{\#} = 0$.

q is a rotation if it is a quaternion (i.e., real, hence $v = 0$). That a spacetime event is expressed as an Hermitian biquaternion rather than a quaternion does not detract from this as $q(t + i\bar{r})q^{\#} = t + iq\bar{r}q^{\#}$.

q is a boost if it is Hermitian, i.e., if u is purely scalar (temporal) and v is purely vector (spatial), i.e., $\bar{u} = 0$ and $v_0 = 0$. It is reasonable to expect that any boost, i.e., of any magnitude in any direction, can be represented in this way because there are three degrees of freedom remaining after the four variables u_0, v_x, v_y, v_z are subject to the constraint $qq^{\#} = 1$.

In fact, it is readily shown that any Lorentz transformation can be expressed as a rotation followed by a transformation by an Hermitian q (i.e., a boost) by explicit construction. Putting,

$$\mu^2 = uu^{\#} = vv^{\#} + 1$$

it follows that $\mu^2 \geq 1$ because $vv^{\#} \geq 0$ is a real quaternion. Hence μ can be taken as real, positive and non-zero. Defining biquaternions $r = \mu^{-1}u$ and $s = \mu - i\mu^{-1}uv^{\#}$ we see that r is a real quaternion and hence represents a rotation, whilst, on the other hand, $s^{*\#} = \mu + i\mu^{-1}vu^{\#} = \mu - i\mu^{-1}uv^{\#} = s$ so that s is Hermitian and so represents a boost. Finally we have,

$$sr = (\mu - i\mu^{-1}uv^{\#})\mu^{-1}u = u - i\mu^{-2}uv^{\#}u = u + i\mu^{-2}vu^{\#}u = u + iv$$

So the rotation r followed by the boost s is equivalent to the initial combined Lorentz transformation.

I think you'll agree this is all very much neater than throwing 4 x 4 matrices around, or dealing with things of the form $exp\left\{-\frac{i}{2}wL_{\alpha\beta}\varepsilon^{\alpha\beta}_{\mu\nu}\right\}$.

Note that if an Hermitian biquaternion, b, transforms under q, i.e., $p \to b \to b' = qbq^{*\#}$ then the complex conjugate Hermitian biquaternion, b^{*}, does not transform under q but under q^{*}, i.e.,

$$b^* \to b'^* = (qbq^{*\#})^* = q^*b^*q^\# \tag{6.1.3}$$

(Incidentally, this means that, for every Hermitian biquaternion, it is necessary to state which transformation applies).

The product of two Hermitian biquaternions is not Hermitian, just as it is not, in general, for complex matrices, because $(pq)^\dagger = q^\dagger p^\dagger = qp$ which will only be the same as pq if they commute, which generally they will not. Nevertheless, the product of two Hermitian biquaternions has a simple transformation as long as they transform oppositely, i.e., one under q and the other under q^*. To put that differently, if Hermitian biquaternions a and b both transform under q then,

$$ab^* \to a'b^{*\prime} = qaq^{*\#}q^*b^*q^\# = qab^*q^\# \tag{6.1.4}$$

This is the simple transformation rule for products ab^* where a and b are both Hermitian biquaternions transforming under q. Note, however, that it is NOT the same as the transformation of an Hermitian biquaternion.

If we write Hermitian biquaternions $a = a_0 + i\bar{a}$ and $b = b_0 + i\bar{b}$ and recalling that $\bar{a}\bar{b} = -\bar{a} \cdot \bar{b} + \bar{a} \times \bar{b}$, where the terms on the RHS refer to the usual 3-vector notation, then we find, if a and b both transform under q,

$$ab^* = (a_0b_0 - \bar{a} \cdot \bar{b}) + \bar{a} \times \bar{b} + i(b_0\bar{a} - a_0\bar{b}) \tag{6.1.5}$$

The transformation $ab^* \to qab^*q^\dagger$ therefore tells us that the usual Lorentz scalar product, $a_0b_0 - \bar{a} \cdot \bar{b}$, is invariant under any Lorentz transform, as it should be. The vector part of ab^*, however, is another matter as this changes under both rotations and boosts.

The equations of classical electromagnetism are highly compact in this biquaternion notation, see Five Square Roots (Appendix L), Ref.[3]. or Lambek, Ref.[4]..

Hermitian biquaternions have a natural expression as exponentials. Any spacetime event at a timelike 4-vector from the origin can be represented as $q = t + i\bar{r}$ where,

$$q = re^{i\theta\hat{n}} = r(cosh(\theta) + i\hat{n}sinh(\theta)) \tag{6.1.6}$$

All future-pointing timelike events wrt the origin can be represented as (6.1.6) for $r > 0$ with $\hat{n}$ ranging over all eight octants (all 4π steradians), and $0 \le \theta < +\infty$. However, to represent past-directed timelike events (with negative scalar part) we require $r < 0$, with $\hat{n}$ again ranging over all eight octants and $0 \le \theta < +\infty$. Note that (6.1.6) can only be timelike because,

(6.1.6) implies: $q^\# q = t^2 - |\bar{r}|^2 = r^2(cosh^2\theta - sinh^2\theta) = r^2 > 0$ (6.1.7)

Spacelike events wrt the origin can be represented as, (6.1.8)

$$q = ire^{\left(\frac{\pi}{2}+i\theta\right)\hat{n}} = ir\left(cos\left(\frac{\pi}{2} + i\theta\right) + \hat{n}sin\left(\frac{\pi}{2} + i\theta\right)\right) = r(sinh(\theta) + i\hat{n}cosh(\theta))$$

Now we have : $q^\# q = t^2 - |\bar{r}|^2 = -r^2$, confirming the spacelike nature of (6.1.8). All spacelike events can be represented by allowing $\hat{n}$ again to range over all eight octants together with $0 \le \theta < +\infty$ in which case r must be allowed to take either sign so as to represent a negative scalar part (time coordinate). This has the advantage of being the same coordinate ranges as for the timelike events.

6.2 Gradnull Defined for Biquaternionic Functions

A variant definition of "gradnull" is deployed for functions which take biquaternionic values. For this purpose we defined an Hermitian gradient operator $D = \partial_t + i\bar{\nabla}$, which is such that $D^{\#} = D^{*} = \partial_t - i\bar{\nabla}$. Hence, we now have,

$$D^{\#}D = \partial_t^2 - \nabla^2 \tag{6.2.1}$$

We define the class of biquaternion-valued functions, $f(t, x, y, z)$, to be b-gradnull by the vanishing of their D "gradient",

$$Df(t, x, y, z) = 0 \tag{6.2.2}$$

We need to distinguish b-gradnull from gradnull because a quaternion-values function is also a special case of a biquaternion function, and $df = 0$ is a different condition from $Df = 0$.

Note that the definition of b-gradnull does require the four derivatives of f wrt the four real variables t, x, y, z to exist. As with gradnull functions, we do NOT write b-gradnull functions as $f(q)$, as if they were functions only of the Hermitian biquaternionic variable $q = t + i\bar{r}$. Instead, b-gradnull functions depend separately on the four real variables t, x, y, z – except that the dependence of the function on these variables is constrained by (6.2.2). In fact, b-gradnull functions cannot be a function of q only because in that case we would have,

$$Df(q) = f'(q)\left[\frac{\partial q}{\partial t} + iI\frac{\partial q}{\partial x} + iJ\frac{\partial q}{\partial y} + iK\frac{\partial q}{\partial z}\right]$$

$$= f'(q)[1 + i^2I^2 + i^2J^2 + i^2K^2] = 4f'(q) \neq 0$$

the only exception being the trivial case that f is a constant.

(6.2.2) immediately implies that quaternionic gradnull functions obey the wave equation because,

$$D^{\#}Df = (\partial_t^2 - \nabla^2)f = 0 \tag{6.2.3}$$

We can also define conjugate-b-gradnull functions by,

$$D^{\#}f(t, x, y, z) = 0 \tag{6.2.4}$$

These also obey the wave equation because,

$$DD^{\#}f = (\partial_t^2 - \nabla^2)f = 0 \tag{6.2.5}$$

Considering an arbitrary biquaternion-valued function of the spatial coordinates x, y, z only, $g(x, y, z)$, it is clear that the function,

$$f(t, x, y, z) = e^{-it\bar{\nabla}}g(x, y, z) \tag{6.2.6}$$

is b-gradnull because,

$$D\left(e^{-it\bar{\nabla}}g\right) = (\partial_t + i\bar{\nabla})e^{-it\bar{\nabla}}g = (-i\bar{\nabla} + i\bar{\nabla})e^{-it\bar{\nabla}}g \equiv 0 \tag{6.2.7}$$

Similarly, considering an arbitrary biquaternion-valued function of the spatial coordinates x, y, z only, $h(x, y, z)$, it is clear that the function,

$$f(t, x, y, z) = e^{it\bar{\nabla}}h(x, y, z) \tag{6.2.8}$$

is conjugate-b-gradnull because,

$$D^{\#}\left(e^{it\bar{\nabla}}h\right) = (\partial_t - i\bar{\nabla})e^{it\bar{\nabla}}h = (i\bar{\nabla} - i\bar{\nabla})e^{it\bar{\nabla}}h \equiv 0 \tag{6.2.9}$$

As well as any function f of the form (6.2.6) being b-gradnull, the reverse also follows if we can assume the Taylor series in t exists. That is, any b-gradnull function can be written like (6.2.6), namely as $f(t,x,y,z) = e^{-t\bar{\nabla}} f(0,x,y,z)$, if all t-derivatives of f exist so we can write,

$$\partial_t f = -i\bar{\nabla} f = -i\left(\bar{\nabla} f_0 + t\bar{\nabla} f_0' + \frac{t^2}{2}\bar{\nabla} f_0'' + \frac{t^3}{3!}\bar{\nabla} f_0''' - \cdots\right) \qquad (6.2.10)$$

where the subscripts 0 denote evaluation at $t = 0$ and the dashes denote derivatives wrt t. Integration of (6.2.10) then gives $f(t,x,y,z) = e^{-it\bar{\nabla}} f(0,x,y,z)$, QED. Similarly, all conjugate-b-gradnull functions can be written like (6.2.8) if the time Taylor series exists.

In a similar manner to the Hamilton-Graves theorem for the 4D Laplace equation, the most general solution to the wave equation is a sum of the two types,

$$(\partial_t^2 - \nabla^2) f = 0 \Leftrightarrow f = e^{-it\bar{\nabla}} g(x,y,z) + e^{it\bar{\nabla}} h(x,y,z) \qquad (6.2.11)$$

where g and h are arbitrary biquaternion-valued functions of the spatial coordinates only. Hence b-gradnull and conjugate-b-gradnull functions are disjoint classes of particular solutions of the wave equation.

The operation of D on a general biquaternion with scalar part u and vector part $\bar{v}$, both of which may be complex, is,

$$D(u + \bar{v}) = (\partial_t u - i\overline{\nabla} \cdot \bar{v}) + (\partial_t \bar{v} + i\overline{\nabla} u + i\overline{\nabla} \times \bar{v}) \qquad (6.2.12)$$

The operation of D on an Hermitian biquaternion $u + i\bar{v}$ where u and $\bar{v}$ are both real, is,

$$D(u + i\bar{v}) = (\partial_t u + \overline{\nabla} \cdot \bar{v}) + (i\partial_t \bar{v} + i\overline{\nabla} u - \overline{\nabla} \times \bar{v}) \qquad (6.2.13)$$

So, despite D and $u + i\bar{v}$ both being Hermitian, $D(u + i\bar{v})$ is not Hermitian in general because of the $\overline{\nabla} \times \bar{v}$ term.

6.3 The b-gradnull Plane Wave

What is the b-gradnull function f with generating function $g = e^{i\bar{k}\cdot\bar{r}}$?

We need to evaluate $e^{-it\bar{\nabla}} e^{i\bar{k}\cdot\bar{r}}$ where $\bar{k} \cdot \bar{r} = k_x x + k_y y + k_z z$.

$\overline{\nabla} e^{i\bar{k}\cdot\bar{r}} = (I\partial_x + J\partial_y + K\partial_z) e^{i\bar{k}\cdot\bar{r}} = (Iik_x + Jik_y + Kik_z) e^{i\bar{k}\cdot\bar{r}} = i\bar{k} e^{i\bar{k}\cdot\bar{r}}$

$\overline{\nabla}^2 e^{i\bar{k}\cdot\bar{r}} = -\nabla^2 e^{i\bar{k}\cdot\bar{r}} = k^2 e^{i\bar{k}\cdot\bar{r}}$ where $k = |\bar{k}|$

$\overline{\nabla}^3 e^{i\bar{k}\cdot\bar{r}} = ik^2 \bar{k} e^{i\bar{k}\cdot\bar{r}}$

$\overline{\nabla}^4 e^{i\bar{k}\cdot\bar{r}} = k^4 e^{i\bar{k}\cdot\bar{r}}$, etc.

$$e^{-it\bar{\nabla}} = 1 - it\bar{\nabla} + \frac{1}{2}(-it\bar{\nabla})^2 + + \frac{1}{3!}(-it\bar{\nabla})^3 + \frac{1}{4!}(-it\bar{\nabla})^4 \ldots$$

$$e^{-it\bar{\nabla}} e^{i\bar{k}\cdot\bar{r}} = \left(1 + t\bar{k} - \frac{1}{2}(tk)^2 - \frac{1}{3!}(t)^3 k^2 \bar{k} + \frac{1}{4!}(tk)^4 \ldots\right) e^{i\bar{k}\cdot\bar{r}}$$

$$= \left(cos(kt) + \hat{k} sin(kt)\right) e^{i\bar{k}\cdot\bar{r}} = e^{\bar{k}t} e^{i\bar{k}\cdot\bar{r}} = e^{i(\bar{k}\cdot\bar{r} - i\bar{k}t)} \qquad (6.3.1)$$

Hence the b-gradnull function which reduces to $e^{i\bar{k}\cdot\bar{r}}$ at $t = 0$ is ***not*** the plane wave $e^{i(\bar{k}\cdot\bar{r} - kt)}$ as one might have expected, but the subtly different beast $e^{i(\bar{k}\cdot\bar{r} - i\bar{k}t)}$ which involves the Hermitian biquaternion, $\bar{k} \cdot \bar{r} - i\bar{k}t$. (Actually the exponent is the anti-Hermitian biquaternion $i\bar{k} \cdot \bar{r} + \bar{k}t$).

Whilst the usual plane wave $e^{i(\bar{k}\cdot\bar{r}-kt)}$ is scalar, the b-gradnull function $e^{i(\bar{k}\cdot\bar{r}-i\bar{k}t)}$ is essentially biquaternionic – but also, of course, a solution of the wave equation.

This presages that any Cauchy-like integral theorems that may be found for b-gradnull functions will not apply to conventional plane waves.

It is immediately clear that the conjugate-b-gradnull function generated by $e^{i\bar{k}\cdot\bar{r}}$ must be $e^{it\bar{\nabla}}e^{i\bar{k}\cdot\bar{r}} = \left(cos(kt) - \hat{k}sin(kt)\right)e^{i\bar{k}\cdot\bar{r}} = e^{i(\bar{k}\cdot\bar{r}+i\bar{k}t)}$ (6.3.2)

Consequently, the sum and difference of (6.3.1) and (6.3.2) give us the usual scalar standing wave solutions $cos(kt)e^{i\bar{k}\cdot\bar{r}}$ and $sin(kt)e^{i\bar{k}\cdot\bar{r}}$ (noting that the factor of $\hat{k}$ is just a constant) and suitable combinations of these give us the usual scalar travelling wave solutions $e^{i(\bar{k}\cdot\bar{r}\pm kt)}$ – and hence the general solution of the wave equation is a linear combination of b-gradnull and conjugate-b-gradnull functions. But what we have discovered is that none of the usual (scalar) wave solutions are b-gradnull or conjugate-b-gradnull.

6.4 The Product of Two b-Gradnull Functions

As for quaternions, it is not generally the case that the product of two b-gradnull functions is b-gradnull. However, the same theorem as before applies: the product FG where both F and G are b-gradnull functions will also be b-gradnull if the generating function of F is scalar, i.e., if $F(t,x,y,z) = e^{-it\bar{\nabla}}f(x,y,z)$ and f is scalar.

Note that f, F and G may be complex, i.e., generally biquaternionic.

The proof is as before. Because G is also b-gradnull it can also be written as $G(t,x,y,z) = e^{-it\bar{\nabla}}g(x,y,z)$ although g need not be scalar. Carrying out the time derivatives, (6.4.1)

$$D(FG) = \left(-i\bar{\nabla}e^{-it\bar{\nabla}}f(x,y,z)\right)\left(e^{-it\bar{\nabla}}g(x,y,z)\right) + \left(e^{-it\bar{\nabla}}f(x,y,z)\right)\left(-i\bar{\nabla}e^{-it\bar{\nabla}}g(x,y,z)\right) + i\bar{\nabla}\left[\left(e^{-it\bar{\nabla}}f(x,y,z)\right)\left(e^{-it\bar{\nabla}}g(x,y,z)\right)\right]$$

Since f is scalar, in the last term above this allows the simple chain rule to hold, i.e.,

$$\bar{\nabla}(fg) = (\bar{\nabla}f)g + f(\bar{\nabla}g) \qquad \text{for scalar } f$$

Using this shows the terms in (6.4.1) to cancel and we conclude that ,

For scalar f: $DF = 0$ and $DG = 0$ implies $D(FG) = 0$ (6.4.2)

7. Integral Theorems for Biquaternionic b-Gradnull Functions

7.1 The Integration Measure

In quaternionic space, i.e., 4D Euclidean space, we could always write the hypersurface integration measure as,

$$d^3S_q = n_t dxdydz + In_x dtdydz + Jn_y dtdxdz + Kn_z dtdxdy \qquad (7.1.1)$$

where $\bar{n}$ is some unit vector in four Euclidean dimensions, $n_t^2 + n_x^2 + n_y^2 + n_z^2 = 1$. In this case d^3S_q is a Euclidean 4-vector.

We are now interested in using the biquaternion formulation to address Minkowski space, and we know that that Hermitian biquaternions have the correct Minkowski norm, (6.1.1). We also know how Hermitian biquaternions transform simply under Lorentz transformations, i.e., as (6.1.2). Hence Hermitian biquaternions are the representation of Minkowski 4-vectors. Hence we require the hypersurface integration measure to also be an Hermitian biquaternion. So under this interpretation,

$$d^3S_{bq} = n_t dxdydz + i\left(In_x dtdydz + Jn_y dtdxdz + Kn_z dtdxdy\right) \qquad (7.1.2)$$

where now either $n_t^2 - \left(n_x^2 + n_y^2 + n_z^2\right) = 1$ for timelike-directed vectors d^3S_{bq}, or $n_t^2 - \left(n_x^2 + n_y^2 + n_z^2\right) = -1$ for spacelike-directed vectors d^3S_{bq}.

Another way of expressing d^3S_{bq} is as follows, (I think)…

Timelike hyperbolae of constant ρ

$$d^3S_{bq} = \pm u\, \rho^3 sinh^2\alpha\ sin\theta\ d\alpha d\theta d\varphi$$

Spacelike hyperbolae of constant ρ

$$d^3S_{bq} = \pm u\, \rho^3 cosh^2\alpha\ sin\theta\ d\alpha d\theta d\varphi$$

where,

u	**Future directed**	**Past directed**
timelike	$cosh(\alpha) + i\hat{n}sinh(\alpha)$	$-cosh(\alpha) + i\hat{n}sinh(\alpha)$
spacelike	$sinh(\alpha) + i\hat{n}cosh(\alpha)$	$-sinh(\alpha) + i\hat{n}cosh(\alpha)$

and where $0 \le \alpha \le \infty$ and $\hat{n}$ is any Euclidean unit vector in 3D (as a quaternion, i.e., $\hat{n} = In_x + Jn_y + Kn_z$), and $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ as usual.

I need to prove the above expressions for d^3S_{bq}, though the rest of this note does not require them.

7.2 The Grad Integral Theorem for Biquaternionic Functions

By analogy with the grad theorem for quaternion functions, (4.2.1), we postulate that the corresponding theorem will hold for biquaternion functions with the quaternion grad, d, replaced by the Hermitian biquaternion grad, D, with the integration measure also understood to be in Minkowski space, as discussed in §7.1. Hence, we hypothesise that the following holds for an arbitrary closed hypersurface in Minkowski space,

$$\Re = \oiiint_{\delta V^4} d^3S_{bq}\, f\,(t,x,y,z) = \int\iiint_{V^4} Df(t,x,y,z)dtdxdydz \qquad (7.2.1)$$

for any biquaternionic function (of suitable smoothness, differentiability, etc).

7.2.1 Proof of the Grad Theorem, (7.2.1), for a Prismatic Hypersurface

In this sub-section we prove (7.2.1) assuming an integration hypersurface, δV_4, which is prismatic. For completeness I repeat the detailed definition of such a surface from §4.2.1, as follows,

- Consider a closed 2-surface δV_3 enclosing a 3D region V_3 of the spatial part of biquaternion space, i.e., the 3-vector part;

- Part of the hypersurface δV_4 is made by extruding δV_3 along the 'time' axis from t_1 to t_2, creating a 4D 'prism' (often called a 'cylinder', though a non-round one) so that V_4 may be identified with $V_3 \otimes [t_1, t_2]$;
- The curved surface of the above 'prism' is turned into a closed hypersurface by adding 'caps' to its ends at times t_1 and t_2,. These 'caps' are simply the spatial 3-volumes V_3 at times t_1 and t_2 respectively.

The biquaternion-valued 3-surface element, $d^3 S_{bq}$, is defined in the obvious manner for each of these regions separately,

- For the end 'caps' the magnitude of $d^3 S_{bq}$ is simply the usual spatial volume element $dxdydz$. Its normal therefore points in the temporal direction, and hence $d^3 S_{bq}$ consists only of a temporal (scalar) part. At t_2 the outward normal is positive, whereas at t_1 the outward normal is negative. Hence $d^3 S_{bq} = d^3 S_0 = dxdydz$ at t_2 but $d^3 S_{bq} = d^3 S_0 = -dxdydz$ at t_1.
- For the curved surface of the hyper-cylinder, the normal to δV_4 is just the normal to δV_3. Hence if we write the normal vector surface element of δV_3 as $d^2 \bar{S}$ then we have simply $d^3 S_{bq} = id^2 \bar{S} dt$, where the factor of i is included to ensure that $d^3 S_{bq}$ is an Hermitian biquaternion (as it happens with zero scalar part on the "cylindrical" surface). As usual, the vector notation must be reinterpreted as a quaternion, replacing the x, y, z unit vectors with I, J, K, and this is essential in the performance of the integral, (7.2.1).

Writing the general biquaternion function as $f = f_0 + \bar{f}$, where both f_0 and $\bar{f}$ could be complex, and expanding the quaternion product, $d^3 S_q f$, gives us,

$$\Re = \oiiint_{\delta V^4} \left[\left(f_0 d^3 S_0 - i\bar{f} \cdot d^2 \bar{S} dt \right) + \left(i f_0 d^2 \bar{S} dt + \bar{f} d^3 S_0 - i\bar{f} \times d^2 \bar{S} dt \right) \right] \quad (7.2.2)$$

Note that the last term is preceded by a minus sign because it actually arises as $+d^2 \bar{S} \times \bar{f} \equiv -\bar{f} \times d^2 \bar{S}$, i.e., the minus sign results from writing the differential $d^3 S_q$ first in the integrand of (7.2.1).

Consider firstly the scalar part of this integral, $\Re_0$. We may convert it to an integral over the 4-volume within V^4 as follows: The first term is only non-zero on the end caps (since $d^3 S_0$ is zero on the curved part of the hypercylinder) and here $d^3 S_0$ is simply the 3-volume element, thus,

$$\begin{aligned} \Re_0(\text{first term}) &= \iiint_{V^3 \text{ at } t_2} f_0 dxdydz - \iiint_{V^3 \text{ at } t_1} f_0 dxdydz \\ &= \iiint_{V^3} \int_{t_1}^{t_2} \frac{\partial f_0}{\partial t} dxdydzdt \end{aligned} \quad (7.2.3)$$

where the last form is simply the 4-volume integral over V^4.

The second term in $\Re_0$ is non-zero only on the curved 'cylindrical' surface. It can also be converted to a 4-volume integral over V^4 by using the divergence theorem,

$$\Re_0(\textit{2nd term}) = -\mathrm{i} \int_{t_1}^{t_2} dt \oiint_{\delta V^3} \bar{f} \cdot d^2 \bar{S} = -i \int_{t_1}^{t_2} dt \iiint_{V^3} \bar{\nabla} \cdot \bar{f} \, dxdydz \quad (7.2.4)$$

(We note that the divergence theorem is equally applicable for complex valued vector functions, simply due to its linearity in $\bar{f}$).

Adding (7.2.3,4) gives,

$$\Re_0 = \int \iiint_{V^4} \left[\frac{\partial f_0}{\partial t} - i\bar{\nabla}\cdot\bar{f}\right] dtdxdydz \tag{7.2.5}$$

We note from (6.2.12) that the integrand in the above is simply the scalar component of Df, and hence consistent with the theorem to be proved, (7.2.1). So far, so good. Now for the 3-vector part:-

We again convert the integrals in (7.2.2) into 4-volume integrals over V^4. The term in d^3S_0 is converted in the same way as (7.2.3), i.e.,

$$\oiiint_{\delta V^4} \bar{f} d^3S_0 \equiv \iiint_{V^3}[\bar{f}(t_2) - \bar{f}(t_1)]\, dxdydz \equiv \int \iiint_{V^4} \frac{\partial \bar{f}}{\partial t} dtdxdydz \tag{7.2.6}$$

The terms in $d^3\bar{S}$ may be converted with the help of the following standard vector integral identities, which we note are equally true for complex-valued functions,

$$\oiint_{\delta V^3} A d^2\bar{S} \equiv \iiint_{V^3} \bar{\nabla} A\, \mathrm{d}^3V \quad \text{and} \quad \oiint_{\delta V^3} \bar{B} \times d^2\bar{S} \equiv -\iiint_{V^3} \bar{\nabla}\times\bar{B}\, \mathrm{d}^3V \tag{7.2.7}$$

which give,

$$\begin{aligned} \oiiint_{\delta V^4} i\left(f_0 d^2\bar{S} - \bar{f}\times d^2\bar{S}\right)dt &= \int_{t_1}^{t_2} dt \oiint_{\delta V^3} i\left(f_0 d^2\bar{S} - \bar{f}\times d^2\bar{S}\right) \\ &= \int_{t_1}^{t_2} dt \iiint_{V^3} i\left(\bar{\nabla} f_0 + \bar{\nabla}\times\bar{f}\right) dxdydz \end{aligned} \tag{7.2.8}$$

Adding (7.2.6,8) gives the 3-vector part of $\Re$ to be,

$$\begin{aligned} \bar{\Re} &= \oiiint_{\delta V^4}\left(i f_0 d^2\bar{S}dt + \bar{f} d^3S_0 - i\bar{f}\times d^2\bar{S}dt\right) \\ &= \int \iiint_{V^4}\left[\frac{\partial \bar{f}}{\partial t} + i\bar{\nabla} f_0 + i\bar{\nabla}\times\bar{f}\right] dtdxdydz \end{aligned} \tag{7.2.9}$$

Comparison with (6.2.12) confirms that the integrand is indeed the 3-vector part of Df. This completes the proof of (7.2.1) for a prismatic surface.

7.2.2 Proof of the Grad Theorem, (7.2.1), for an Arbitrary Closed Hypersurface

The most elegant way is to use the Stokes-Cartan derivation – see the accompanying paper.

7.3 Proposed Cauchy-type Integral for b-Gradnull Biquaternionic Functions

(7.2.1) establishes that b-gradnull biquaternionic functions (i.e., $Df = 0$) are such that,

$$\oiiint_{\delta V^4} d^3S_{bq}\, f\,(t,x,y,z) = 0 \tag{7.3.1}$$

Following the proof of the theorem (5.3.30) for quaternionic gradnull functions, we postulate that the integral,

$$\oiiint_{\delta V^4} d^3S_{bq} e^{-i(t-t_0)\bar{\nabla}}\left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right) f(t,x,y,z) \tag{7.3.2}$$

where the closed surface of integration contains the event (t_0, x_0, y_0, z_0), will be some non-zero constant times $f(t_0, x_0, y_0, z_0)$. We already know that (7.3.2) will be independent of the integration surface because the integrand is b-gradnull everywhere except at (t_0, x_0, y_0, z_0) by virtue of (6.4.2). Consequently, by shrinking the integration surface onto (t_0, x_0, y_0, z_0) it is clear that $f(t_0, x_0, y_0, z_0)$ factors out of the integral. Moreover, as the resulting integration of the function $e^{-i(t-t_0)\bar{\nabla}}\left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right)$ is continues to be independent of δV^4 we can evaluate it on any convenient surface.

Before doing so we find the function $e^{-i(t-t_0)\bar{\nabla}}\left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right)$ explicitly by carrying out the derivatives, closing following §5.2.

7.3.1 Evaluation of the Singular 'Projection' Function

In this section we evaluate $\widetilde{H} = e^{-it\bar{\nabla}}(x/r^4)$ following §5.2 by simply inserting factors of i and powers thereof. Hence, using (5.2.1-8) we find,

which lead to, (7.3.3)

$$e^{-it\bar{\nabla}}\left(\frac{x}{r^4}\right) = \frac{x}{r^4}\left[1 + 2\frac{t^2}{r^2} + 3\frac{t^4}{r^4} + 4\frac{t^6}{r^6} + \ldots\right] \text{ -i}\frac{It}{r^4}\left[1 + \frac{2}{3}\frac{t^2}{r^2} + \frac{3}{5}\frac{t^4}{r^4} + \frac{4}{7}\frac{t^6}{r^6} + \ldots\right] + \text{i}\frac{4xt\bar{r}}{r^6}\left[1 + \frac{3}{3}\frac{t^2}{r^2} + \frac{6}{5}\frac{t^4}{r^4} + \frac{10}{7}\frac{t^6}{r^6} + \ldots\right]$$

The infinite sums in (7.3.3) are given by, with $\xi = t/r$,

$$[first] = \frac{1}{(1-\xi^2)^2} \tag{7.3.4}$$

$$[sec\,o\,nd] = \frac{1}{2}\left[\frac{1}{(1-\xi^2)} + \frac{1}{\xi}tanh^{-1}\xi\right] \tag{7.3.5}$$

$$[third] = \left[\frac{5-3\xi^2}{8(1-\xi^2)^2} + \frac{3}{8\xi}tanh^{-1}\xi\right] \tag{7.3.6}$$

Equs (5.2.11,12) are easily checked using,

$$tanh^{-1}\xi = \xi + \frac{\xi^3}{3} + \frac{\xi^5}{5} + \frac{\xi^7}{7} + \ldots \tag{7.3.7}$$

Hence we have,

$$\widetilde{H}(t,x,y,z) \equiv e^{-it\bar{\nabla}}\left(\frac{x}{r^4}\right) = \frac{x}{(t^2-r^2)^2} - iI\frac{t}{2r^2}\left\{\frac{1}{r^2-t^2} + \frac{1}{rt}tanh^{-1}\left(\frac{t}{r}\right)\right\} + 4ixt\frac{\bar{r}}{r^4}\left\{\frac{5r^2-3t^2}{8(t^2-r^2)^2} + \frac{3}{8rt}tanh^{-1}\left(\frac{t}{r}\right)\right\} \tag{7.3.8}$$

Although we have rederived this from first principles, (7.3.8) is (as expected) just (5.2.15) with t replaced by it. In the same way, just as we have seen that (5.2.15) is consistent, in the quaternionic case, with,

(quaternionic, $q = t + \bar{r}$) $$e^{-t\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{-q^{\#}}{|q|^4} = \frac{-t+\bar{r}}{(r^2+t^2)^2} \tag{5.2.20}$$

Replacing t with it this becomes $e^{-it\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = \frac{-it+\bar{r}}{(r^2+t^2)^2} = -i\frac{(t+i\bar{r})}{(r^2-t^2)^2}$. So that,

(biquaternionic, $q = t + i\bar{r}$) $$e^{-it\bar{\nabla}}\left(\frac{\bar{r}}{r^4}\right) = -i\frac{q}{|q|^4} \tag{7.3.9}$$

will be consistent with (7.3.8), despite appearances.

7.3.2 Evaluation of the Biquaternionic Cauchy Integral on a Long Cylinder

In this section we carry out the integral

$$\Re = \oiiint_{\delta V^4} d^3S_{bq}e^{-i(t-t_0)\bar{\nabla}}\left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right) \tag{7.3.10}$$

on the prismatic surface defined in §7.3.1 in the limit that the end caps are pushed to $t \to \pm\infty$ and the 'cylinder' surface is the 2-sphere of radius ρ. The calculation is largely as in §5.3.2 except for the one contribution which is non-zero where there is a subtlety required in interpreting the resulting integral in order to render it finite. Recall that (7.3.10) is actually the integral,

$$\Re = \oiiint_{\delta V^4} d^3S_{bq} e^{-i(t-t_0)\bar{\nabla}} \left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right) \quad (7.3.11)$$

where $\widetilde{H}$ is given explicitly by (7.3.8).

End Caps

The end caps are the ordinary 3D spatial volume integral within a sphere of radius ρ at $t = t_0$ minus that at $t = -t_0$. The first term in (7.3.8), the scalar component, is zero on the end caps as $t_0 \to \infty$. The same is true for the first terms within each of the two {...}. For $t_0 \to \pm\infty$ the $tanh^{-1}\left(\frac{t}{r}\right)$ terms are $\pm i\pi/2$. As the integrand is equal and opposite on the two end caps the two ends do *not* cancel but add. That leaves us with the requirement to integrate spatially over the function,

$$i\left(-I\frac{\pi}{4r^3} + \frac{3\pi x\bar{r}}{4r^5}\right)$$

This is just the same as the end cap integral in §5.3.2 except for the factor of i and so we already know the end caps integrate to zero.

Cylindrical Surface

Here we have,

$$d^3S_{bq} = idtd^2S_q = idt\,\hat{r}r^2sin\theta\,d\theta d\varphi$$

$$= i(Isin\vartheta cos\varphi + Jsin\theta sin\varphi + Kcos\theta)dt\,r^2sin\theta\,d\theta d\varphi$$

Hence, considering firstly the scalar term in the integrand, (7.3.8), we require,

$$\int i(Isin\vartheta cos\varphi + Jsin\theta sin\varphi + Kcos\theta)\frac{rsin\theta cos\varphi}{(t^2-r^2)^2}dt\,r^2sin\theta\,d\theta d\varphi$$

The φ integral kills the J and K terms and we are left with,

$$iI\int_0^{\pi} sin^3\theta d\theta \int_0^{2\pi} cos^2\varphi d\varphi \int_{-\infty}^{+\infty}\frac{r^3}{(t^2-r^2)^2}dt = \frac{4\pi}{3}iI\int_{-\infty}^{+\infty}\frac{r^3}{(t^2-r^2)^2}dt \quad (7.3.12)$$

We will come back to the evaluation of the integral $\int_{-\infty}^{+\infty}\frac{r^3}{(t^2-r^2)^2}dt$ later as this involves careful handling to avoid being divergent due to the singularities that now occur at $t = \pm r$. We did not have this difficulty when working in Euclidean quaternion space.

We now show that, as in §5.3.2, the two vector parts of (7.3.8) both integrate to zero.

For the first of the vector terms this is immediately clear because it contains no angular dependence, and so inevitably integrates to zero over the vectorial integration measure $d^2S_q \propto \hat{r}$.

For the second vector term, its product with d^2S_q involves $\hat{r}^2 = -1$, which eliminates the angular dependence of those factors. The remaining angular-dependent terms are $x.sin\theta d\theta d\varphi = rsin^2\theta cos\varphi d\theta d\varphi$ which is killed by the φ integral.

Hence, the required integral, (7.3.11) reduces to just (7.3.12). We now need to face the issue of the singularity in that integral over t.

Evaluation of (7.3.12)

The indefinite integral is,

$$\int \frac{r^3}{(t^2-r^2)^2} dt = \int \frac{dx}{(x^2-1)^2} = \frac{x}{2(1-x^2)} + \frac{1}{4} log\left(\frac{1+x}{1-x}\right) \qquad (7.3.13)$$

as may be checked by differentiating the RHS. However, the integrand is singular at $x = \pm 1$. Nevertheless, if we take (7.3.13) literally and substitute the limits we get,

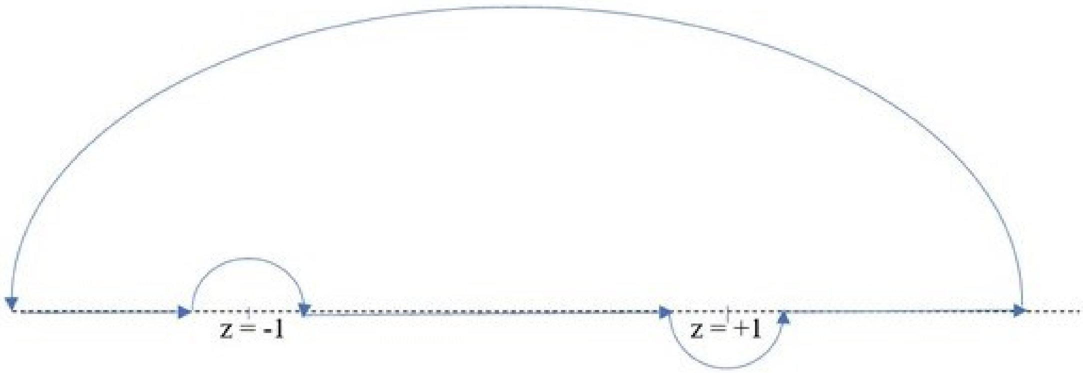
$$\int_{-\infty}^{\infty} \frac{r^3}{(t^2-r^2)^2} dt = \int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2} = \left(\frac{x}{2(1-x^2)} + \frac{1}{4} log\left(\frac{1+x}{1-x}\right)\right)\Bigg|_{-\infty}^{\infty} = \frac{1}{4} log(-1) - \frac{1}{4} log(-1)$$

This is horribly ill-defined (and looks to be zero as written) but $log(-1)$ could be taken as $-i\pi$ at the upper limit ($+\infty$) but $i\pi$ at the lower limit ($-\infty$), hence giving an integral of $-i\pi/2$. There are two ways of getting at this same result which are not so obviously suspect.

The best way is to replace x with a variable z which can be complex, and reinterpret the integral as a contour integral in the complex plane. There is still an ambiguity in the result of the integration depending on the choice of contour with respect to the two poles at $z = \pm 1$. However, we choose to complete the closed contour in the upper half plane and to include the pole at $z = +1$ but to exclude the pole at $z = -1$. Hence we reinterpret the integral to be,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2} \rightarrow \oint_C \frac{dz}{(z^2-1)^2} \qquad (7.3.14)$$

where the contour C is,



where it is understood that the upper half-circle is pushed to infinity, and hence contributes nothing to the contour integral. The contour integral in this limit does, therefore, become the desired real integral – but subject to our choice of which pole to include within the contour and which to exclude.

The poles are of second order. Hence the residue at $z = 1$ is given by,

Residue: $\frac{d}{dz}\left[(z-1)^2 \frac{1}{[(z-1)(z+1)]^2}\right]\Big|_{z=1} = -2(z+1)^{-3}|_{z=1} = -\frac{1}{4}$ (7.3.15)

The contour integral is thus $2\pi i$ times this residue, hence $-i\pi/2$, thus reproducing the heuristic result, above. According to the choice of which poles to include or exclude, however, we could equally well assign a value of $+i\pi/2$ (including only the pole at $z = -1$ since its residue is +1/4) or zero if either both or neither pole are included.

However, we note that the above choice of contour, in assigning the value $-i\pi/2$ to the integral $\int_{-\infty}^{\infty} \frac{r^3}{(t^2-r^2)^2} dt$, means that (7.3.12) gives the final result $\Re = \frac{2}{3}\pi^2 I$, exactly the same as the quaternion gradnull integral, (5.3.30-32). Hence we have,

$$\oiiint_{\delta V^4} d^3 S_{bq} e^{-i(t-t_0)\bar{\nabla}} \left(\frac{x-x_0}{|\bar{r}-\bar{r}_0|^4}\right) f(t,x,y,z) = \frac{2}{3}\pi^2 I f(t_0,x_0,y_0,z_0) \qquad (7.3.16)$$

Similar equations apply in y, z and J, K, of course.

The key integration result, $\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2} = -i\pi/2$ can also be obtained by considering the principal value of the integral added to the contributions form the small half-circles around the poles, as follows.

The principle value is defined as,

$$\mathbb{P}\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2} = 2\,\mathbb{P}\int_{0}^{\infty} \frac{dx}{(x^2-1)^2} = 2Lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \frac{dx}{(x^2-1)^2} + \int_{1+\varepsilon}^{\infty} \frac{dx}{(x^2-1)^2}\right) \qquad (7.3.17)$$

Using the explicit indefinite integral, (7.3.13), the extreme limits produce zero whilst the $1 \pm \varepsilon$ limits provide equal contributions, giving the principal value to be,

$$Lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right) - i\frac{\pi}{2} \qquad (7.3.18)$$

which, of course, is divergent. The sign of the finite term in (7.3.18) requires arbitrarily assigning $log(-1)$ to be $-i\pi$ rather than $+i\pi$ which it could equally well be.

The lower half-circle contour around the pole $z = 1$ is evaluated by setting $z - 1 =$ so that part of the integral becomes,

$$\int \frac{dz}{(z^2-1)^2} = \int \frac{i\varepsilon e^{i\theta} d\theta}{4\varepsilon^2 e^{2i\theta}} = \frac{i}{4\varepsilon}\int_{\pi}^{2\pi} e^{-i\theta} d\theta = \frac{i}{4\varepsilon} ie^{-i\theta}\Big|_{\pi}^{2\pi} = -\frac{1}{4\varepsilon}(1 - -1) = -\frac{1}{2\varepsilon}$$

The upper half-circle contour around the pole $z = -1$ is evaluated by setting $z + 1 =$ so that part of the integral becomes,

$$\int \frac{dz}{(z^2-1)^2} = \int \frac{i\varepsilon e^{i\theta} d\theta}{4\varepsilon^2 e^{2i\theta}} = \frac{i}{4\varepsilon}\int_{\pi}^{0} e^{-i\theta} d\theta = \frac{i}{4\varepsilon} ie^{-i\theta}\Big|_{\pi}^{0} = -\frac{1}{4\varepsilon}(1 - -1) = -\frac{1}{2\varepsilon}$$

So the two half-circles together give a contribution $-\frac{1}{\varepsilon}$. Adding this to the principal value, (7.3.18), we find that the divergent parts cancel, leaving just $-i\pi/2$, in agreement with the method of residues.

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