A Quaternion Analogue of Cauchy's Theorem

RAWB (Original circa 2001; note added 21/4/2015)

One of the things which make functions of a complex variable so powerful to use is Cauchy's theorem. If a function f(z) is analytic in a region Γ of the complex plane, then $\oint f(z)dz = 0$ when the integral is taken around any closed contour within Γ. In addition, the value of the function at any point z_0 within Γ can be found, rather miraculously, in terms of the values remote from z_0 by the integral $\frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0} dz = f(z_0)$. For quaternion valued functions of a quaternion valued variable, can we devise a similar integral? This will be required to yield the value of an "analytic" function at an arbitrary point within an arbitrary 4D region in terms of an integral over a closed hypersurface. But first we must define appropriately what is meant by "analytic" for quaternion functions.

A quaternion-valued function, f, of t, x, y, z is called q-analytic in a region of (t, x, y, z) space if $df = 0$ everywhere within the region, where $d \equiv \partial_t + \overline{\nabla}$ and $\overline{\nabla} \equiv i\partial_x + j\partial_y + k\partial_z$.

By multiplying by d* it follows that q-analytic functions obey the 4D Laplace equation, $\nabla^2_{4D} f = \left(\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right) f = 0$.

The associated q*-analytic functions, g, obey d*g = 0 and are also solutions to $\nabla^2_{4D} g = 0$. Graves-Hamilton (1858) showed that q-analytic and q*-analytic functions together provide the general solution to the 4D Laplace equation, and can be written in terms of a 'generating function' as,

$$f(t,x,y,z) = e^{-t\overline{\nabla}}\left(\tilde{f}(x,y,z)\right) \quad \text{and} \quad g(t,x,y,z) = e^{+t\overline{\nabla}}\left(\tilde{g}(x,y,z)\right) \tag{1}$$

where $\tilde{f}$ and $\tilde{g}$ are, in general, quaternion valued, but are independent of t.

NB: f and g are *not* functions of the quaternion 'position vector' $q = t + ix + jy + kz$ alone. This is a crucial point in which quaternion analytic functions differ from complex analytic functions, and it renders them far less tractable (as does their lack of commutativity).

Defining a quaternion-valued element of 3-surface as d^3S_q, it is simply shown that for any q-analytic function, f, the closed-surface integral around an arbitrary 4-volume V is zero, i.e.,

$$\oiiint_{\delta V^4} d^3S_q f(t,x,y,z) = 0 \tag{2}$$

The relationship with the divergence theorem is obvious. However, it appears that this result is not entirely trivial since the integral $\oiiint_{\delta V^4} f(t,x,y,z) d^3S_q$ is *not* in general zero.

An analogue of Cauchy's theorem with a 'pole term' would be,

$$\oiiint_{\delta V^4} d^3S_q H(t - t_0, x - x_0, y - y_0 z - z_0).f(t,x,y,z) = K.f(t_0, x_0, y_0, z_0) \tag{3}$$

for an arbitrary surface δV^4 provided that it enclose the 'pole' at (t_0, x_0, y_0, z_0), for some constant K, and for some function H which is the analogue of $\frac{1}{z - z_0}$. I claim to have found such a function. It is,

$$H(t,x,y,z) = e^{-t\overline{\nabla}}\left(\frac{x}{r^4}\right) \qquad \text{and} \qquad K = \frac{2\pi^2 i}{3} \tag{4}$$

where r is the 'spatial' radius, i.e. $r^2 = x^2 + y^2 + z^2$. Obviously, (3) also holds with the 'x' in H replaced by y or z, as long as the 'i' in K is replaced by j and k respectively.

The magnitude of K is seen to be simply the volume of a hypersphere, divided by 3 (equal shares for each spatial coordinate).

Omitting the 'x' in H would result in an integrand which is 'spatially' spherically symmetric for an infinitesimal sphere centred on (t_0, x_0, y_0, z_0), and hence would integrate to zero. This illustrates the role of the 'x' in H, namely to provide, in conjunction with d^3S_q, terms which are even over the integration range and hence produce a non-zero integral.

A key feature of H is that it is q-analytic [by virtue of being of the form given in (1)] and its generating function is real. This is crucial. Unlike functions of a complex variable, the product of two q-analytic functions is not, in general, q-analytic. However, it is easily shown that if f_1 and f_2 are both q-analytic, *and* f_1 has a real generating function, then the product f_1f_2 is q-analytic. [NB: the function f_2f_1 is still not q-analytic]. Consequently Hf is q-analytic by construction for an arbitrary q-analytic f.

The importance of this is that (2) then shows that the integral in (3) is independent of the surface δV^4 integrated over – provided that we do not move the 'pole' (t_0, x_0, y_0, z_0) in or out of it.

Details of the proof are available on request.

Note added by RAWB 21/4/15:-

I now realise that (3) is closely related to Fueter's Theorem (Refs.1,2). The reason is that,

$$e^{-t\overline{\nabla}}\left(-\frac{\overline{r}}{r^4}\right) \equiv \frac{1}{|q|^2 q} \tag{5}$$

so multiplying (3) by $-i$ and adding the similar equations in y and z gives,

$$\oiiint_{\delta V^4} \frac{1}{|q-q_0|^2(q-q_0)} d^3S_q f(t,x,y,z) = 2\pi^2 f(t_0,x_0,y_0,z_0) \tag{6}$$

which is Fueter's Theorem. However my theorem, (3), appears to be the stronger result because, whilst (6) follows immediately from (3) the reverse is not obvious.

References

[1] Fueter, R: "Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen." *Comment. math. Helv.* **7**, 307-330 (1935).

[2] Fueter, R: "Uber die analytische Darstellung der regularen Funktionen einer Quaternionenvariablen." *Comment. math. Helv.* **8**, 371-378 (1936).