Quantum Tunnelling

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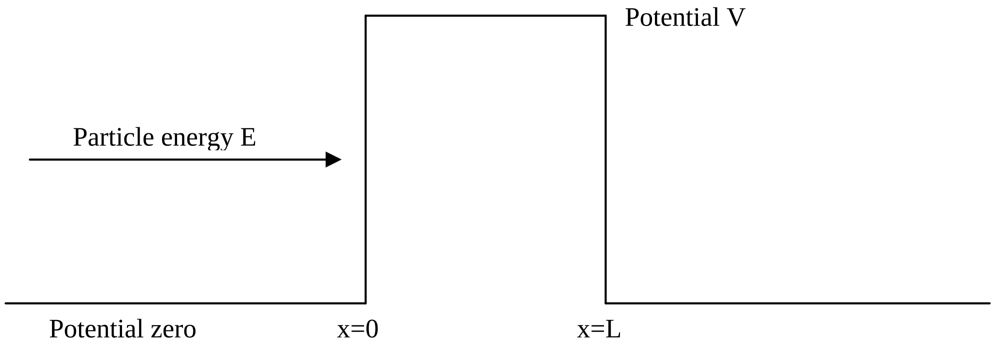
A potential barrier looks like this,



In classical physics, a particle with energy, E, less than the height, V, of the potential barrier has no chance at all of penetrating the barrier. But in quantum mechanics, there is a chance that it will. The probability that a particle penetrates the barrier reduces as either height, V, or the width, L, of the barrier increases. In this note, the fraction of incident particles penetrating the barrier is calculated using the appropriate solutions of the Schrodinger equation.

To the left and right of the barrier the Schrodinger equation is that for a free particle, with zero potential, i.e.,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \qquad (1)$$

The solutions to this are the plane waves, $\psi = Ae^{ikx} + Be^{-ikx}$. The first term represents a wave travelling to the right, and the second term a wave travelling to the left. The wavenumber, k, is given in terms of the energy by,

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad (2)$$

This follows simply by substituting the plane wave solution into (1). We take the incoming wave on the left of the barrier to have unit amplitude (the amplitude is arbitrary, it just sets the absolute number of particles hitting the barrier per second). But there will also be a reflected wave on the left of the barrier, so the totalk wavefunction on the left is,

$$\psi_L = e^{ikx} + \mathrm{R}e^{-ikx} \qquad (3)$$

where R is the amplitude of the reflected wave. If no particles could tunnel through the barrier, then $|R| = 1$, meaning that all the particles get reflected. If some particles tunnel through, then $|R| < 1$.

On the right of the barrier the transmitted wave must be travelling to the right (there is no wave incoming from the right), so that the wavefunction on the right is,

$$\psi_R = Te^{ikx} \qquad (4)$$

where T is the amplitude of he transmitted wave. The fraction of the particles hitting the barrier which manage to tunnel through is $|T|^2$, and the rest are reflected (so we must get $|R|^2 + |T|^2 = 1$).

What does the wavefunction look like inside the barrier? Here there is a (constant) potential, V, so the Schrodinger equation is,

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi = E\psi \tag{5}$$

The general solution inside the barrier is thus,

$$\psi_B = Ae^{\kappa x} + Be^{-\kappa x} \tag{6}$$

This looks very like the plane wave solution, but notice that there is no ‘i’ in the exponents. These are real exponents and so represent exponentially decaying waves. Also, the value of the parameter κ differs from *k*. Substitution of the solution into (5) gives,

$$\kappa = \frac{\sqrt{2m(V-E)}}{\hbar} \tag{7}$$

Note that we have assumed that the barrier potential is greater than the particle energy, $V > E$, so that κ is a real number. [*If* $V < E$ *then* κ *would be imaginary and the wavefunction would be a normal travelling wave*].

So, we now have three different expressions, (3), (4) and (6), for the wavefunction in the three different regions, and these involve a total of four unknown constants, *R*, *T*, *A* and *B*. How do we determine what these constants are? The answer is that the wavefunction must be continuous between the regions. Moreover, the first derivative (i.e., the gradient) of the wavefunction must be continuous too. Consider firstly the front face of the barrier at *x* = 0. We require,

$$\psi_L(x=0) = \psi_B(x=0) \quad \text{and} \quad \left.\frac{\partial \psi_L}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_B}{\partial x}\right|_{x=0} \tag{8}$$

Substituting (3) and (6) into (8) gives,

$$1 + R = A + B \quad \text{and} \quad ik - ikR = \kappa A - \kappa B \tag{9}$$

Now consider the other face of the barrier at $x = L$. Here we require,

$$\psi_R(x=L) = \psi_B(x=L) \quad \text{and} \quad \left.\frac{\partial \psi_R}{\partial x}\right|_{x=L} = \left.\frac{\partial \psi_B}{\partial x}\right|_{x=L} \tag{10}$$

Substituting (4) and (6) into (10) gives,

$$Ae^{\kappa L} + Be^{-\kappa L} = Te^{ikL} \quad \text{and} \quad A\kappa e^{\kappa L} - B\kappa e^{-\kappa L} = ikTe^{ikL} \tag{11}$$

Hennce, in (9) and (11) we have four equations will suffice to find the four constants *R*, *T*, *A* and *B*. By multiplying the first of equations (11) by κ and then adding or subtracting the two equations (11) we get *A* and *B* in terms of *T*,

$$2\kappa e^{\kappa L} A = (\kappa + ik)e^{ikL}T \quad \text{and} \quad 2\kappa e^{-\kappa L} B = (\kappa - ik)e^{ikL}T \tag{12}$$

Substituting the first of equations (9) into the second gives, after re-arranging,

$$\left(1+\frac{\kappa}{ik}\right)A+\left(1-\frac{\kappa}{ik}\right)B=2 \qquad (13)$$

Substituting into (13) for A and B in terms of T from (12) gives,

$$\left\{\left(1+\frac{\kappa}{ik}\right)\left(\frac{\kappa+ik}{2\kappa}\right)e^{(ik-\kappa)L}+\left(1-\frac{\kappa}{ik}\right)\left(\frac{\kappa-ik}{2\kappa}\right)e^{(ik+\kappa)L}\right\}T=2 \qquad (14)$$

Or,

$$\left\{\frac{(\kappa+ik)^2}{2i\kappa k}e^{-\kappa L}-\frac{(\kappa-ik)^2}{2i\kappa k}e^{+\kappa L}\right\}T=2e^{-ikL} \qquad (15)$$

This gives us the exact value of the transmission coefficient, $|T|^2$, i.e., the fraction of the incident particles which quantum tunnel through the barrier,

$$|T|^2=\left|\frac{4k\kappa}{(\kappa+ik)^2e^{-\kappa L}-(\kappa-ik)^2e^{+\kappa L}}\right|^2 \qquad (16)$$

The exact result simplifies if we assume that the barrier is high or wide enough so that $\kappa L>1$, and hence that we can neglect $e^{-\kappa L}$ compared with $e^{+\kappa L}$. (16) then becomes,

$$|T|^2\approx\left[\frac{4k\kappa}{(\kappa^2+k^2)}\right]^2e^{-2\kappa L} \qquad (17)$$

Using (2) and (7) this can be written in terms of the energy and the barrier height as,

$$|T|^2\approx 16\frac{E}{V}\left(1-\frac{E}{V}\right)e^{-2\kappa L} \qquad (18)$$

Recall that this approximation is valid only if $\kappa L>1$, which ensures that (18) yields a fraction.

If the barrier is high (in which case κ is large), or the barrier is wide (in which case L is large), then the exponential in (18) ensures that only a very small proportion of the incident particles tunnel through the barrier.

But, even if the barrier height, V, is a lot greater than the particles' energy, E, there may still be a high proportion of particles penetrating the barrier if the barrier is sufficiently narrow (small L).