Is Quantum Mechanics Contradictory?

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I expose here an awkward feature of quantum mechanics which is rather brushed under the carpet in traditional treatments. If a pair of complementary observables obey the canonical commutation relation, $[\hat{P}, \hat{Q}] = i\xi$, for some constant number ξ, and if both observables take values from a discrete set, $\{p_i\}$ and $\{q_i\}$, i.e. they have discrete eigenvalues, then a contradiction results from the usual formulation.

Proof:

Put $\hat{Q}|q_i\rangle = q_i|q_i\rangle$. Consider the matrix element of the commutator between two q-states,

$$\langle q_i|(\hat{P}\hat{Q} - \hat{Q}\hat{P})|q_j\rangle = \langle q_i|(\hat{P}q_j - q_i\hat{P})|q_j\rangle = (q_j - q_i)\langle q_i|\hat{P}|q_j\rangle = i\xi\delta_{ij} \qquad \text{(QM4.1)}$$

For i = j this results in the contradiction that the RHS is non-zero but the LHS is zero.

How can we resurrect quantum mechanics from this stark and elementary catastrophe? I have never seen a discussion of it (except in the context of P and Q being interpreted as angular momentum and angular position, in Peierls, ????, and here it was regarded as a pathology only of this case due to the multiple valued nature of the angular position coordinate, $Q = \theta$).

When P and Q are interpreted as a Cartesian momentum and position, the usual representation is in terms of a continuum coordinate, $Q = x$. In this case the Kronecker delta in (QM4.1), when divided by the difference in position coordinates, metamorphoses into a Dirac delta function, so that (QM4.1) becomes,

$$\langle x'|\hat{P}|x\rangle = i\xi\frac{\delta_{xx'}}{(x - x')} \propto i\xi\delta(x - x') \qquad \text{(QM4.2)}$$

So, the usual property of the Dirac delta function being zero when $x \neq x'$ but infinite when $x = x'$ is consistent with this expression. In this way QM salvages the incipient disaster – apparently.

But note what a heavy price is paid for this resurrection. QM is a theory which is all about discreteness. It is formulated in terms of finite algebra. And yet for the theory to avoid a serious inconsistency it is obliged to adopt the continuum. Discreteness at any level would reintroduce the contradiction of (QM4.1). This is physically dubious, to put it mildly, since the continuum exists only in mathematics not in the physical world.

Moreover, the generalised functions (distributions) which the adoption of the Dirac delta function imposes on QM has serious repercussions. It is perhaps little appreciated that the divergences of quantum field theory (QFT) can be traced to this source. This fact is disguised in QFT by the convention of working in momentum space. The divergences then appear as divergent k-integrals. But in configuration space these integrals result from products of distributions – which are undefined (i.e. divergent) when evaluated at $x = x'$.

So, my contention is that there is something rotten in the state of Quantum Mechanics. The most obvious manner in which to rectify the problem (short of discarding the whole Hilbert space formalism) is to modify the commutator, $\left[\hat{P}, \hat{Q}\right] = i\xi$. Instead of such commutators being constants, we may postulate an operator-valued expression, $\left[\hat{P}, \hat{Q}\right] = i\hat{\xi}$. This might avoid the inconsistency if $\hat{\xi}$ does not commute with either of $\hat{P}$ or $\hat{Q}$. This means that $\hat{\xi}$ would have different eigenvectors and hence that $\langle q_i | \hat{\xi} | q_j \rangle$ would not have to be proportional to δ_{ij}. To avoid the inconsistency it would be necessary to ensure that $\langle q_i | \hat{\xi} | q_i \rangle = 0$ [and also $\langle p_i | \hat{\xi} | p_i \rangle = 0$] for all i. In other words, $\hat{\xi}$ would have to have all zeros down the principal diagonal in both the q-basis and the p-basis. Because the trace of an operator is invariant under a unitary transformation, this implies that the sum of the eigenvalues of $\hat{\xi}$ must be zero. An example is, of course, provided by the angular momentum operators, $\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z$.

So, can a version of quantum mechanics with only discrete eigenvalues, and no nasty divergent objects like the Dirac delta function, be formulated successfully via $\left[\hat{P}, \hat{Q}\right] = i\hat{\xi}$? I don't know. However, assuming that the uncertainty relation is still required to be $\Delta_x \cdot \Delta_p \geq \hbar/2$, then we can deduce that $\left|\langle \psi | \hat{\xi} | \psi \rangle\right| \geq \hbar/2$ for any state $|\psi\rangle$. (This would bear checking – I have not done so carefully enough yet). This implies that all the eigenvalues of $\hat{\xi}$ must have $|\lambda| \geq \hbar/2$. It is possible that the operator $\hat{\xi}$ is saved from degenerating into the constant $\hbar/2$ by having some eigenvalues which are greater than this lower bound, i.e., $|\lambda| > \hbar/2$ for some λ. Does this produce a distinct but workable quantum mechanics? Does it offer any advantage?