The Uncertainty Principle

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1. Introduction

It is telling that, in the initial compilation of these QM notes, I left out the uncertainty principle altogether. This would have been unthinkable a few decades ago, since the uncertainty principle was regarded as the defining essence of quantum mechanics. Not only that, but the uncertainty principle is closely related to the non-zero commutator between complementary observables, and hence exists at the heart of the algebraic structure of the theory. In truth, the uncertainty principle is still a central concept in QM, and is a defining distinction between QM and classical physics. This has not changed. But I now perceive that the role of the commutator has weakened.

The Dirac prescription for obtaining a quantum theory from its classical counterpart is to replace the classical Poisson brackets between observable with the commutator, $[\hat{A}, \hat{B}]$. In the case of complementary observables, e.g. Cartesian momentum and position, the commutator is set to $i\hbar$, i.e. $[\hat{P}_x, \hat{x}] = i\hbar$. This has entered the lexicon as "canonical quantisation", i.e. the standard process for obtaining a quantum theory from its classical forebear. The reason why this correspondence has become to seem less important might be that familiarisation with quantum theories has lead to a confidence in raising quantum models directly, without the prop of a classical precursor. Personally (and I could be wrong) I begin to doubt that commutator relations like $[\hat{P}_x, \hat{x}] = i\hbar$ are even strictly correct in general (see Part 4 of these notes, QM4). I am not alone. Streater (2003) has purely mathematical reasons for dismissing the Dirac quantisation procedure. Nor is scepticism about "quantisation" anything new (see Schroer, 2003).

Nevertheless, if we pick an arbitrary pair of Hermetian operators acting on a given Hilbert space, in almost all cases they will not commute. This means they will have different eigenvectors[1]. But quantum mechanics holds that an observable projects the state of the system onto one of its eigenstates upon being measured. It follows that a system cannot be simultaneously in a definite state of two non-commuting observables. This is the essence of the uncertainty principle.

However, the uncertainty principle as usually stated goes further. It considers a quantification of the uncertainty in each of a pair of complementary observables and

[1] The proof is simple. Suppose they had the same eigenvectors. Each operator can be expressed in the form $A = U\Lambda_A U^+$ and $B = U\Lambda_B U^+$, where U is the matrix of orthogonal eigenvectors, and is the same for the two operators by assumption. The matrices Λ_A and Λ_B are diagonal and consist of the eigenvalues, which may differ. Hence we have,
$AB = U\Lambda_A U^+ U\Lambda_B U^+ = U\Lambda_A\Lambda_B U^+ = U\Lambda_B\Lambda_A U^+ = U\Lambda_B U^+ U\Lambda_A U^+ = BA$, so that operators with the same eigenvectors necessarily commute. QED. Note that Λ_A and Λ_B commute by virtue of being diagonal.

states that the product of these individual uncertainties must exceed a certain specified minimum. This (quantified) uncertainty principle follows from commutation relations such as $[\hat{P}_x, \hat{x}] = i\hbar$, and leads to $\Delta x \cdot \Delta P_x \geq \frac{\hbar^2}{4}$. This is proved below.

2. Proof of the Uncertainty Relation

We assume two Hermetian operators with commutator $[\hat{P}, \hat{x}] = i\hbar$, one interpretation of which is one Cartesian component of momentum and position. We consider the modified operators defined as the difference of these operators from some arbitrary, but constant, value,

$$\Delta\hat{x} = \hat{x} - x_0 \quad \text{and} \quad \Delta\hat{P} = \hat{P} - P_0 \qquad \text{(QM3.2.1)}$$

where x_0 and P_0 are just numerical values. Hence, the mean square deviation of the observables $\hat{x}$ and $\hat{P}$ from these constant values are given by,

$$\left\langle \Delta x^2 \right\rangle = \left\langle \psi \middle| \Delta\hat{x}^2 \middle| \psi \right\rangle \quad \text{and} \quad \left\langle \Delta P^2 \right\rangle = \left\langle \psi \middle| \Delta\hat{P}^2 \middle| \psi \right\rangle \qquad \text{(QM3.2.2)}$$

where the arbitrary state of the system is labelled ψ. This state can be expanded in some arbitrary orthonormal basis as $|\psi\rangle = c_i |\phi_i\rangle$, where summation over repeated subscripts will be assumed hereafter. Note that the basis states $\{|\phi_i\rangle\}$ need not be eigenstates of either $\hat{x}$ or $\hat{P}$. The action of these operators on the basis can be written,

$$\Delta\hat{x}|\phi_i\rangle = a_{ij}|\phi_j\rangle \quad \text{and} \quad \Delta\hat{P}|\phi_i\rangle = b_{ij}|\phi_j\rangle \qquad \text{(QM3.2.3)}$$

Substitution into (QM3.2.2) yields,

$$\left\langle \Delta x^2 \right\rangle = |\overline{u}|^2 \quad \text{and} \quad \left\langle \Delta P^2 \right\rangle = |\overline{v}|^2 \qquad \text{(QM3.2.4)}$$

where the complex-valued vectors $\overline{u}$ and $\overline{v}$ are defined by,

$$u_j \equiv c_i a_{ij} \quad \text{and} \quad v_j \equiv c_i b_{ij} \qquad \text{(QM3.2.5)}$$

But Schwartz's inequality tells us that, for any complex values vectors in any number of dimensions,

$$|\overline{u}|^2 |\overline{v}|^2 \geq |\overline{u}^* \cdot \overline{v}|^2 \qquad \text{(QM3.2.6)}$$

[This inequality is obvious. It says that the product of the squared-lengths of two vectors must be greater than or equal to the square of their dot product. Since the dot product is the product of the lengths of the vectors times the cosine of the included angle, this is clearly true – the equality holding only when the vectors are parallel and the included angle is zero. The only thing that complicates this is the complex nature of the vectors. However, this has no effect on the LHS, which involves only the moduli of the vectors. On the RHS the effect of the phase difference between the two vectors can only lead to partial cancellations, and hence further reduces the magnitude of the RHS. QED].

Schwartz's inequality suggests we consider the expectation value of the product of $\Delta\hat{x}$ and $\Delta\hat{P}$, i.e. $\langle\psi|\Delta\hat{x}\Delta\hat{P}|\psi\rangle$, which, on substitution of (QM3.2.3,5) does indeed turn out to equal $|\bar{v}^* \cdot \bar{u}| = |\bar{u}^* \cdot \bar{v}|$. Hence, Schwartz's inequality gives,

$$\langle\Delta x^2\rangle\langle\Delta P^2\rangle \geq |\langle\psi|\Delta\hat{x}\Delta\hat{P}|\psi\rangle|^2 \qquad \text{(QM3.2.7)}$$

The operator appearing on the RHS is not Hermetian, because $\hat{x}$ and $\hat{P}$ do not commute. It may be re-written,

$$\Delta\hat{x}\Delta\hat{P} = \frac{1}{2}[\Delta\hat{x},\Delta\hat{P}] + \frac{1}{2}\left(\Delta\hat{x}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{x}\right) = i\frac{\hbar}{2} + \frac{1}{2}\left(\Delta\hat{x}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{x}\right) \qquad \text{(QM3.2.8)}$$

The last operator on the RHS is now Hermetian, by virtue of its symmetry and the Hermetian nature of $\hat{x}$ and $\hat{P}$. Consequently it has real eigenvalues and hence a real expectation value wrt any state. In contrast, the first (purely numerical) term on the RHS is imaginary. It follows that, when taking the absolute square of the RHS on (QM3.2.7) there is no cross-product between the two terms, and we have,

$$\langle\Delta x^2\rangle\langle\Delta P^2\rangle \geq \frac{\hbar^2}{4} + \frac{1}{4}\left|\langle\Delta\hat{x}\Delta\hat{P} + \Delta\hat{P}\Delta\hat{x}\rangle\right|^2 \qquad \text{(QM3.2.9)}$$

But the second term on the RHS is positive, so we get,

$$\langle\Delta x^2\rangle\langle\Delta P^2\rangle \geq \frac{\hbar^2}{4} \qquad \text{(QM3.2.10)}$$

which is the usual expression of the uncertainty relation.

The term we have dropped from the RHS of (QM3.2.9) can be zero, and hence (QM3.2.10) is the strongest general form of the expression.

Note that (QM3.2.10) has been derived for arbitrary values of the constants x_0 and P_0. It is usual to interpret these constants as the expectation values of the operators $\hat{x}$ and $\hat{P}$, i.e.,

$$x_0 = \langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle \quad \text{and} \quad P_0 = \langle \hat{P} \rangle = \langle \psi | \hat{P} | \psi \rangle \qquad \text{(QM3.2.11)}$$

so that $\langle \Delta\hat{x} \rangle = \langle \psi | \Delta\hat{x} | \psi \rangle = 0$ and $\langle \Delta\hat{P} \rangle = \langle \psi | \Delta\hat{P} | \psi \rangle = 0$. It is easily seen that adopting $x_0 = \langle \hat{x} \rangle$ and $P_0 = \langle \hat{P} \rangle$ leads to the smallest mean square deviations, $\langle \Delta x^2 \rangle$ and $\langle \Delta P^2 \rangle$, and hence provides the strongest form of uncertainty relation.