QM12: The Quantum Zeno Paradox: A Watched Pot Never Boils

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Suppose a system starts in a state $|1\rangle$, which is perhaps an eigenstate of some undisturbed Hamiltonian, $\hat{H}_0$. Now suppose we apply some influence to the system, described by an interaction Hamiltonian. $\hat{H}_I$. This has the potential to change the state of the system (heating the cold pot could make it boil). Indeed, if we left the combined effects of $\hat{H}_0 + \hat{H}_I$ alone to do their work, the state of the system *would* change (the pot *would* boil). But what happens if we repeatedly measure the state of the system – or, to be more precise, if we measure its energy by applying $\hat{H}_0$? What happens if we continually 'watch the pot'?

In classical physics this would make no difference, of course. An observation may be made without disturbing the system. But not so in quantum mechanics. It turns out that if we measure the system's energy state often enough, then its state will never change: a watched quantum pot never boils.

The most general time-dependent state of the system can be written in terms of the eigenstates, $|j\rangle$, of the free Hamiltonian as,

$$|t\rangle = \sum_j a_j(t) \exp\left\{-i\frac{E_j t}{\hbar}\right\}|j\rangle \tag{1}$$

We shall be concerned only with changes over small periods of time, so we can use first order perturbation theory secure in the knowledge that this will be exact in the limit. Suppose at t=0 the system is in eigenstate j=1, i.e., $a_1(0)=1$, $a_j(0)=0$ for $j>1$. First order perturbation theory[1] tells us that the time dependent expansion coefficients are given by,

For $j>1$
$$\frac{\partial a_j}{\partial t} = \langle j|\hat{H}_I|1\rangle \exp\left\{i\frac{(E_j - E_1)}{\hbar}t\right\} \tag{2}$$

Again since we are concerned only with the limit of very short times, it suffices to consider $\hat{H}_I$ to be time-independent, without loss of generality, apart from the fact that it is turned on only at t=0. We can then integrate (2) explicitly, giving,

For $j>1$
$$a_j(t) = i\frac{\langle j|\hat{H}_I|1\rangle}{E_j - E_1}\left(1 - \exp\left\{i\frac{(E_j - E_1)}{\hbar}t\right\}\right) \tag{3}$$

Hence,
$$|a_j(t)|^2 = 2\left|\frac{\langle j|\hat{H}_I|1\rangle}{E_j - E_1}\right|^2\left(1 - \cos\left\{\frac{(E_j - E_1)}{\hbar}t\right\}\right) \tag{4}$$

We are only claiming that this is universally correct in the limit of short times, so more properly we write,

For $j>1$ and $LIM(t \to 0)$:
$$|a_j(t)|^2 = \left|\frac{\langle j|\hat{H}_I|1\rangle}{E_j - E_1}\right|^2\left\{\frac{(E_j - E_1)}{\hbar}t\right\}^2 = \left|\frac{\langle j|\hat{H}_I|1\rangle}{\hbar}t\right|^2 \tag{5}$$

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[1] See the other notes on web page "Introduction to Quantum Mechanics"

At any time that a measurement is made, the probability that the system will be found to be in the original state, $|1\rangle$, is $P = |a_1(t)|^2$, which is given by,

$$P = |a_1(t)|^2 = 1 - \sum_{j>1} |a_j(t)|^2 = 1 - \left(\sum_{j>1} \left| \frac{\langle j | \hat{H}_I | 1 \rangle}{\hbar} \right|^2 \right) t^2 \equiv 1 - \chi t^2 \qquad (6)$$

Here χ is defined as the constant $\chi \equiv \sum_{j>1} \left| \frac{\langle j | \hat{H}_I | 1 \rangle}{\hbar} \right|^2$. Consequently, if we leave it long enough before measuring the state of the system, we may find that it is no longer in the initial state – because $P < 1$. If you don't watch the pot, it will eventually boil.

But see what happens if the system is measured N times at equal intervals during the period t, i.e., at intervals of t/N. The probability that the system remains in state $|1\rangle$ after the first measurement is,

$$P_1 = 1 - \chi \left(\frac{t}{N} \right)^2 \qquad (7)$$

Assuming that the system is in state $|1\rangle$ after the first measurement, the probability that it is still in state $|1\rangle$ after the second measurement is also given by (7). So the overall probability of being in state $|1\rangle$ after two measurements is,

$$P_2 = P_1 \times P_1 = \left[1 - \chi \left(\frac{t}{N} \right)^2 \right]^2 \qquad (8)$$

Similarly, after N measurements, the probability that the state remains $|1\rangle$ is,

$$P_N = P_1^N = \left[1 - \chi \left(\frac{t}{N} \right)^2 \right]^N \qquad (9)$$

Now as we let N become very large, so we are making a very large number of measurements in the fixed time period, t, this probability tends to unity:

$$LIM(N \to \infty): P_N \to 1 \qquad (10)$$

The reason can be traced to the quadratic dependence on time in (6), and hence the quadratic denominator in N in (9).

Physically, the repeated measurements continually knock the system back into its initial state, neutralising the influence of the disturbance which would otherwise cause its state to change.