Semi-Infinite Plate Subject to an In-Plane Point Force

This note solves the following linear elasticity problem: a semi-infinite plate whose boundary lies along the x-axis is subject to a point force in the y-direction acting at the point (0,1), i.e., at a unit distance from the boundary of the plate and perpendicular to it.

Part 1: Point Load acting on an Infinite Plate

We initially consider an infinite plate, i.e. with no boundary, subject to the same y-force at point (0,1). In this case the load is reacted at infinity. A bit of inspired guesswork suggests that the complex Airy function which solves this problem is,

$$\phi = \mathrm{Im}\{x \log(z - i)\} \tag{1}$$

where z is the complex coordinate $z = x + iy$. The point load is therefore applied at $z = i$. This function clearly satisfies the Airy equation $\nabla^4 \phi = 0$, since it is a linear function times a function analytic everywhere except at the point load, $z = i$. Evaluating the stresses from this Airy function gives,

$$\sigma_x = \phi_{,yy} = \mathrm{Im}\left\{\frac{x}{(z-i)^2}\right\} = -\frac{2}{r}\sin\theta\cos^2\theta \tag{2}$$

$$\sigma_{xy} = -\phi_{,xy} = \mathrm{Im}\left\{\frac{1-y}{(z-i)^2}\right\} = -\frac{2}{r}\sin^2\theta\cos\theta \tag{3}$$

$$\sigma_y = \phi_{,xx} = \mathrm{Im}\left\{\frac{2}{z-i} - \frac{x}{(z-i)^2}\right\} = -\frac{2}{r}\sin^3\theta \tag{4}$$

In (2-4) r, θ are polar coordinates developed around the point load, i.e. defined by $z - i = r\exp\{i\theta\}$. We can now check that (1) does indeed correspond to an applied point load in the y-direction. To do so, consider a circle centred on the point load, i.e., at some fixed radius r. Now find the total force acting across this boundary and corresponding to the stresses given in (2-4). We find, for the total x-force,

$$F_x = \oint\left[\sigma_x \cos\theta + \sigma_{xy}\sin\theta\right]\cdot r d\theta = -2\int_{-\pi}^{+\pi}\left[\sin\theta\cos^3\theta + \sin^3\theta\cos\theta\right]\cdot d\theta = 0 \tag{5}$$

The integral is clearly zero because both its terms are odd functions. Note that, because the stresses in (2-4) vary with radius as 1/r, the radial dependence cancels in (5). Hence, there is zero net x-force on every circle centred on the point load.

In a similar manner we find the y-force to be,

$$\begin{aligned} F_y &= \oint\left[\sigma_{xy}\cos\theta + \sigma_y \sin\theta\right]\cdot r d\theta = -2\int_{-\pi}^{+\pi}\left[\sin^2\theta\cos^2\theta + \sin^4\theta\right]\cdot d\theta \\ &= -2\int_{-\pi}^{+\pi}\left[\sin^2\theta\right]\cdot d\theta = -2\pi \end{aligned} \tag{6}$$

Hence there is a net y-force of -2π acting on the material within the circle, i.e., a net y-force of +2π acting on the plate. Because the radial dependence cancels, this same net y-force applies irrespective of the size of circle considered. In particular, it applies if the circle is shrunk to a point. Hence, (1) corresponds to an applied point load of magnitude 2π in the y-direction, as claimed. QED.

Part 2: Point Load acting on a Semi-Infinite Plate

We now introduce a symmetry plane along the x-axis, i.e., on the plane y = 0. It is clear that this is physically equivalent to an infinite plate with two equal and opposite point loads acting at $z = i$ and $z = -i$, i.e., at the points on the y-axis with $y = \pm 1$. Consequently this problem is solved as the linear superposition of (1) and the equivalent solution with the loading point shifted to $z = -i$ and the sign reversed. Hence, the solution is given by the Airy function,

$$\phi = \mathrm{Im}\{x\log(z-i) - x\log(z+i)\} = \mathrm{Im}\left\{x\log\frac{z-i}{z+i}\right\} \tag{7}$$

If correct, this should produce zero shear on the y = 0 plane. With a bit of algebra, we find the complete solution for the stresses to be, (8)

$$\sigma_x = \phi_{,yy} = \mathrm{Im}\left\{\frac{x}{(z-i)^2} - \frac{x}{(z+i)^2}\right\} = \frac{4x^2\left[\left(x^2-y^2+1\right)^2 - 4x^2y^2\right] + 16x^2y^2\left(x^2-y^2+1\right)}{\left[\left(x^2-y^2+1\right)^2 + 4x^2y^2\right]^2}$$

$$\sigma_{xy} = -\phi_{,xy} = -\mathrm{Im}\left\{\frac{-y+1}{(z-i)^2} - \frac{-y-1}{(z+i)^2}\right\} = \frac{8xy\left(x^2+y^2-1\right)\left(x^2-y^2+1\right)}{\left[\left(x^2-y^2+1\right)^2 + 4x^2y^2\right]^2} \tag{9}$$

$$\sigma_y = \phi_{,xx} = \mathrm{Im}\left\{\frac{2}{z-i} - \frac{2}{z+i} - \frac{x}{(z-i)^2} + \frac{x}{(z+i)^2}\right\} = \mathrm{Im}\left\{\frac{2}{z-i} - \frac{2}{z+i}\right\} - \sigma_x$$
$$= \frac{4\left(x^2-y^2+1\right)}{\left[\left(x^2-y^2+1\right)^2 + 4x^2y^2\right]} - \sigma_x \tag{10}$$

Hence, the shear is zero on y = 0 as required. Symmetry also requires that the shear and the x-stress be zero on the y-axis, i.e., for x = 0. This also is obeyed by (8,9). On the x-axis, at unit distance from both point loads, the stresses reduce to simply,

$$\sigma_x = \frac{4x^2}{\left(1+x^2\right)^2} \qquad \text{and} \qquad \sigma_y = \frac{4}{\left(1+x^2\right)^2} \tag{11}$$

Recall that these stresses refer to a load of 2π. Hence, for a load of magnitude F, the stresses on y = 0 are,

$$\sigma_x = \frac{2Fx^2}{\pi\left(1+x^2\right)^2} \qquad \text{and} \qquad \sigma_y = \frac{2F}{\pi}\frac{1}{\left(1+x^2\right)^2} \tag{12}$$

These stresses are plotted in Figure 1. For a load at some arbitrary distance, D, from the symmetry plane (or boundary), replace *x* by the physical *x* coordinate normalised by D, i.e., $x \to x/D$.

By integrating the y-stress along the x-axis we can check that the total reaction force is F, as it should be. This follows from the integral $\int_{-\infty}^{+\infty} \frac{dx}{\left(1+x^2\right)^2} = \frac{\pi}{2}$, which is simply proved by the substitution $x = \tan\alpha$. **<u>QED</u>**.