Fatigue Crack Growth: Why The Paris Law?

Last Update: 16 March 2008

Very small cracks can grow by fatigue under the influence of a cyclic plastic zone which would be present even in the absence of the crack, e.g. the zone of plastic strain near a notch. But when a crack is large enough, the crack tip field creates its own cyclic plastic zone. The crack will then grow by fatigue under the influence of its self-generated cyclic plasticity. It is reasonable to expect the fatigue crack growth per cycle to be related to the stress intensity factor range, ΔK, or the post-yield equivalent, ΔJ. Empirically, structural metals are found to have fatigue growth rates $\frac{da}{dN} \propto \Delta K^m$, where m generally lies between 3 and 4. Why does m take a value of about this magnitude? Why is m not 1 or 10, for example? This is explained here.

In order for fatigue to cause crack growth, energy must be deposited near the crack tip. "Deposited" means that the energy must be irreversibly absorbed in the form of damage to the material. The mechanism of fatigue crack growth is energy deposition via reversing cyclic plasticity.

On first loading, a plastic zone is formed around the crack tip. For simplicity we shall consider only the case of small scale yielding, where the plastic zone around the crack tip is small compared with the remaining ligament. When the load is removed, most of the material within the crack tip plastic zone simply unloads down the elastic line. In other words, most of the plastic zone cycles elastically after first yielding[1]. This is because, whilst the material within the plastic zone clearly exceeds the yield stress, most of this material will be at less than twice the yield stress (when calculated elastically). Roughly speaking, it is only the material whose elastic stress exceeds twice the yield stress which will fall within the zone of reversing plasticity. Only this material is responsible for absorbing energy and hence driving fatigue crack growth.

The crudest model of material behaviour is this: assume that within the zone of cyclic plasticity the stress and the strain both take some uniform "yield" values, σ_0 and ε_0, and reverse to minus these values on unloading. Further assume that the size of the cyclic plastic zone can be estimated from the LEFM fields by requiring $\Delta\sigma = 2\sigma_0 = \frac{\Delta K}{\sqrt{2\pi r_Z}}$, so that the cyclic plastic zone has size,

$$r_Z = \frac{\Delta K^2}{8\pi\sigma_0^2} \qquad (1)$$

The energy deposited per unit volume is just $2\sigma_0\varepsilon_0$. Assuming the plastic zone is a circle centred on the crack tip, the energy deposited per cycle is thus,

[1] We are also assuming for simplicity that all the cycles are of the same magnitude

$$\text{energy deposited per cycle} = \frac{\varepsilon_0 t \Delta K^4}{32\pi\sigma_0^3} \qquad (2)$$

where t is the thickness (i.e. the length of crack front considered). Assuming it requires a fixed amount of energy to grow the crack by a unit increment of area (G_c), the crack growth per cycle is therefore,

$$\frac{da}{dN} = \frac{\varepsilon_0 \Delta K^4}{32\pi\sigma_0^3 G_c} \qquad (3)$$

This is a fatigue crack growth law of Paris type. It should certainly not be used in assessments! The only purpose of this derivation is to rationalise the typical magnitude of the exponent m, in this case m = 4.

For fun, though, we can ask if the magnitude of the coefficient in Equ.(3) does align roughly with data from fatigue crack growth tests. In R66 Section 10, a Paris law with m = 4 is given for CMn steels in the short-transverse orientation. For illustration assume a typical CMn yield stress of 250 MPa at a strain of 0.125%, and hence E = 200 GPa. Assume a best-estimate fracture toughness of 160 MPa$\sqrt{}$m, which implies Gc ~ 0.13 MPa.m. The coefficient in (3) thus evaluates to ~5 x 10^{-12} for growth in metres and ΔK in MPa$\sqrt{}$m. This is in remarkably good agreement with the value in R66 at ambient temperature (4 x 10^{-12}). Probably this is a lucky fluke, but it supports the credibility of the mechanism.

One of the especially naïve features of the above derivation (though there are many) is the assumption of a uniform "yield" stress and strain within the cyclic plastic zone. A more realistic approach is to assume the HRR fields apply. Recalling that J/σ has units of length, the stress and strain fields vary with distance, r, from the crack tip as,

$$\sigma \propto \sigma_0 \left(\frac{J}{\sigma_0 r} \right)^{\frac{1}{n+1}} \qquad \text{and} \qquad \varepsilon \propto \varepsilon_0 \left(\frac{J}{\sigma_0 r} \right)^{\frac{n}{n+1}} \qquad (4)$$

where power law hardening is assumed, $\frac{\varepsilon}{\varepsilon_0} = \left(\frac{\sigma}{\sigma_0} \right)^n$. Instead of a uniform energy deposition per unit volume we now have a deposition which is more intense near the crack tip,

$$\text{energy deposited per unit volume} \propto \sigma\varepsilon = \frac{\varepsilon_0 J}{\sigma_0 r} \propto \frac{\Delta K^2}{r} \qquad (5)$$

To find the total energy deposited we therefore need to integrate (5) over the volume of the cyclic plastic zone. The volume element is $2\pi rt.dr$. Hence we get,

$$\text{energy deposited per cycle} \propto t r_z \Delta K^2 \qquad (6)$$

where r_z is the cyclic plastic zone size, again assumed circular and centred on the crack tip. This is found by equating the PYFM stress to twice yield, i.e.,

$$\sigma = 2\sigma_0 \propto \sigma_0 \left(\frac{J}{\sigma_0 r} \right)^{\frac{1}{n+1}} \qquad (7)$$

but this once again gives $r_z \propto \Delta J \propto \Delta K^2$, as in the LEFM case, Equ(1). Substituting this into Equ.(6) gives,

$$\text{energy deposited per cycle} \propto \Delta K^4 \qquad (8)$$

So the exponent m = 4 in the Paris law persists when PYFM crack tip fields are assumed.

Why, for most structural steels of interest, do fatigue tests tend to give m closer to 3 than 4? This is left as an exercise for the reader.