The Physical Origin of the Lorentz Transformation in the Equivalence of Inertial Observers

It is easy to get lost in the algebraic elegance of Lie groups and so forth and to forget just what the Lorentz transformation means. There is nothing mysterious or surprising about transformations which mix space and time. Consider two observers in relative uniform motion in the x-direction. Suppose their x-origins coincide at time t = 0. Observer S' measures an object to be at a coordinate x', whereas observer S places it at his coordinate x. If S' is moving with velocity u in the x-direction wrt S then, in pre-relativity physics, the transformation is simply $x' = x - ut$. So there we have it: the mixing of space and time in the transformation. Motion tends to do that, you know. Before relativity came along, time was regarded as universal. So our two observers would (we thought) have agreed on the time of any event. The complete transformation is thus,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{1}$$

This non-relativistic transformation connecting inertial observers is known as a Galilean transformation. Prior to relativity emerging, (1) was regarded as so self-evident that it was not then dignified with so grand a sounding title as a "transformation".

> We wish to explore what the most general linear relationship between the (x,t) coordinates of two inertial observers could be, constrained only by the principle that all inertial observers are equivalent.

But firstly we address a couple of preliminaries.

Why confine attention to a linear transformation? The answer to this is that we can always confine attention to a small region of spacetime, so that we are really interested in the transformation of some small spacetime displacement (dx, dt). In this case, we could have an arbitrary non-linear transformation of coordinates, $x' = f(x,t)$ and $t' = g(x,t)$, but this then gives,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial t} \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{2}$$

As long as we confine attention to a small region within which the gradients of f and g do not vary significantly, then this becomes simply,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{3}$$

i.e., a constant linear transformation. More technically, we may be dealing with a curved spacetime manifold, but we are addressing the properties of the tangent space at a point. So confining attention to (3) with constant elements does not represent a restriction as long as we are content with purely *local* transformations.

The second preliminary is: what is implied regarding the matrix in (3) if the relative velocity of the observers is u? The relative velocity is the velocity of the origin of S'

as observed by S. The origin of S' is at $dx' = 0$ so (3) gives $a \cdot dx + b \cdot dt = 0$, which gives $\frac{dx}{dt} = u = -\frac{b}{a}$. Hence we can eliminate b and write (3) as,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} a & -au \\ d & e \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{4}$$

OK, we are now ready to explore what further constraints on the form of the transformation in (4) follow from the principle that all inertial observers are equivalent. Note that the parameters a, d, e in (4) can depend upon the velocity. The equivalence of S and S' means that we could equally write (4) with the S coordinates on the LHS and the S' coordinates on the RHS, so long as we change the relative velocity from u to $-u$. The velocity dependence is shown explicitly below,

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \begin{pmatrix} a(-u) & a(-u) \cdot u \\ d(-u) & e(-u) \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix} \tag{5}$$

However by inverting (4) we also have,

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \frac{1}{ae + adu} \begin{pmatrix} e & au \\ -d & a \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix} \tag{6}$$

where it is understood in (6) that the parameters are all evaluated at u not $-u$. But the matrices in (5) and (6) must be equal since (5) and (6) hold for arbitrary spacetime displacements. In particular, equating the $_{12}$ components gives,

$$a(-u) = \frac{a}{ae + adu} \tag{7}$$

where it is again understood in (7) that the parameters on the RHS are evaluated at u not $-u$.

Now if we consider an interval in frame S with $dt = 0$ we have from (4) that $dx' = a(u) \cdot dx$. And if we consider a space-like interval in frame S' with $dt' = 0$ we have from (5) that $dx = a(-u) \cdot dx'$. But the equivalence of observers means that these two situations should be equivalent, so that it must be that $a(-u) \equiv a(u)$, i.e., a is an even function of u. Using this in (7) gives,

$$a(e + du) = 1 \tag{8}$$

Equating the other components between (5) and (6) then gives,

$$e = a \quad \text{and} \quad d(-u) = -d(u) \tag{9}$$

Now consider a third observer, S'', moving with velocity u' wrt S'. We have,

$$\begin{pmatrix} dx'' \\ dt'' \end{pmatrix} = \begin{pmatrix} a' & -a'u' \\ d' & a' \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix} \tag{10}$$

where the dashes on a and d refer to the same functions of velocity as in (4) but evaluated at u'. Substituting (4) into (10) gives,

$$\begin{pmatrix} dx'' \\ dt'' \end{pmatrix} = \begin{pmatrix} a'a - a'u'd & -a'a(u + u') \\ d'a + da' & -d'au + a'a \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{11}$$

But this must have the same form as a transformation direct from S'' to S, and so must equal,

$$\begin{pmatrix} dx'' \\ dt'' \end{pmatrix} = \begin{pmatrix} a'' & -a''u'' \\ d'' & a'' \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{12}$$

where the double dash represents quantities evaluated at the relative velocity of S'' wrt S, i.e., u''. We have made no assumption regarding how to find u'' in terms of u and u'. Comparing (11) and (12) shows that the diagonal elements in (11) must be equal, so we require,

$$a'a - a'u'd = -d'au + a'a \tag{13}$$

This gives,

$$\frac{a'u'}{d'} = \frac{au}{d} = \kappa \tag{14}$$

Equ.(14) must hold for arbitrary velocities u and u' and hence this combination of the functions must be independent of velocity, i.e., a universal constant, κ, as written. The transformation, (4), therefore must be,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = a \begin{pmatrix} 1 & -u \\ u/\kappa & 1 \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{15}$$

for some universal constant, κ, to be determined. Finally, (8) now becomes,

$$a\left(a + \frac{a}{\kappa}u^2 \right) = 1 \tag{16}$$

Hence,

$$a = \frac{1}{\sqrt{1 + \frac{u^2}{\kappa}}} \tag{17}$$

So finally the most general transformation consistent with the principle of the equivalence of inertial observers is,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \frac{1}{\sqrt{1 + \frac{u^2}{\kappa}}} \begin{pmatrix} 1 & -u \\ u/\kappa & 1 \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{18}$$

The Galilean transformation, (1), is regained if we take $\kappa \to \infty$. More generally the meaning of the universal constant, κ, can be elucidated as follows. By dividing the space equation in (18) by the time equation, we get,

$$v' = \frac{v - u}{1 + \frac{uv}{\kappa}} \tag{19}$$

where $v = \frac{dx}{dt}$ and $v' = \frac{dx'}{dt'}$ are the velocities of some particle as seen by the two observers. For convenience of exposition, replace u by $-u$ in (19), so that S' is moving in the negative x-direction and sees a positive-going particle moving faster than does S. (19) becomes, with all positive numbers u, v, v',

$$v' = \frac{v+u}{1-\dfrac{uv}{\kappa}} \tag{20}$$

Assuming that $\kappa > 0$, (20) implies that there are finite speeds *u* and *v* with $uv = \kappa$, for which S' sees the particle moving infinitely fast, $v' \to \infty$. This establishes that infinite relative speeds can occur in this case, and hence we can consistently consider arbitrarily large *u* in the transform (18). But if we consider $u^2 >> \kappa$ then (18) implies $dx' \to -\sqrt{\kappa} \cdot dt$ and $dt' \to dx / \sqrt{\kappa}$, because the diagonal components become very small in this limit. But physically it makes no sense for a pure time interval to be transformed into a pure spatial interval. Physically there is a preferred direction of time (future pointing). Such a transformation would induce a preferred spatial direction. Consequently we can rule out $\kappa > 0$.

We conclude that $\kappa < 0$ and write $\kappa = -c^2$, so that (20) becomes,

$$v' = \frac{v+u}{1+\dfrac{uv}{c^2}} \tag{21}$$

Now the opposite happens. No infinite speeds are possible. Instead the form of (21) ensures that as $u \to c$ for a finite $v < c$ we get $v' \to c$. Similarly, as $v \to c$ at finite $u < c$ we also get $v' \to c$. With this interpretation, *c* acts as a limiting speed. The most general transformation consistent with the principle of the equivalence of inertial observers is thus,

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -u \\ -u/c^2 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{22a}$$

where,

$$\gamma = \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} \tag{22b}$$

Equs.(22) are the familiar form of the Lorentz transformation.

The important point is that the form of the Lorentz transformation and the existence of a limiting speed has been deduced from the principle of the equivalence of inertial observers.

Can the Galilean assumption, that the limiting speed is infinite, be ruled out by purely theoretical reasoning? Ultimately experiment is the arbiter. The Michelson–Morley experiment showed directly that the speed of light did not vary for different motions of its source, thus having exactly the characteristics of a limiting speed in accord with (22). However the Michelson–Morley result did not automatically lead to relativity theory. It took Einstein to recognise its significance in terms of a spacetime transformation. And the route he took was via the transformation properties of the electromagnetic field equations. Thus the identification of the limiting speed with that of electromagnetic wave propagation through the vacuum was a theoretical insight.