Normal Modes: 'Orthogonal', But Not As We Know It

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I shall use the term 'orthogonal' in the following sense: real vectors $\overline{u}$ and $\overline{v}$ are orthogonal if & only if $\overline{u}^T\overline{v} = \sum_i u_i v_i = 0$; a matrix O is orthogonal iff $O^T \equiv O^{-1}$; an orthogonal transformation of a matrix M is $M' = O^T M O$ where O is an orthogonal matrix. These conform to standard mathematical usage of the term 'orthogonal'. In particular, the orthogonality of two real vectors is identified with their inner (dot) product being zero. If u_i and v_i are Cartesian components, albeit in general in an N-dimensional space, this shows that the orthogonality of two real vectors can be identified with the vectors being perpendicular in the usual way.

It is shown below that the normal modes of a structure are ***not*** orthogonal, despite the usual claims. This utterly confounded me for some time. Theorems follow:-

(1) If a matrix M can be diagonalised by an orthogonal transformation, O, then O is composed of the normalised eigenvectors of M.

Proof: Put $M\overline{\psi}_i = \lambda_i \overline{\psi}_i$ and $\Lambda = O^T M O = \text{diagonal}$ (Equ.1)

Then, $O^T M O O^T \overline{\psi}_i = \lambda_i O^T \overline{\psi}_i$ (Equ.2)

or $\Lambda \overline{\psi}'_i = \lambda_i \overline{\psi}'_i$ where, $\overline{\psi}'_i = O^T \overline{\psi}_i$ (Equ.3)

Since Λ is diagonal, its normalised eigenvectors, $\overline{\psi}'_i$, are just $(1,0,0,0\ldots)^T$, $(0,1,0,0,\ldots)^T$, $(0,0,1,0,0,\ldots)^T$ etc and the on-diagonal elements are the eigenvalues, λ_i. From the inverse of the second of Equs.3 it then follows that the transform O is composed of the eigenvectors of M, i.e.,

$\overline{\psi}_i = O\overline{\psi}'_i$ hence $\overline{\psi}_i = i^{th}$ column of O hence $O = (\overline{\psi}_1 \overline{\psi}_2 \overline{\psi}_3 \ldots)$ (Equ.4)

QED.

<u>Corollary A</u>: There is only one (unique) <u>orthogonal</u> matrix which diagonalises any given matrix M (up to arbitrary ordering of the columns), i.e. the eigenvectors.

<u>Corollary B</u>: A matrix which can be diagonalised by an orthogonal transform must have orthogonal eigenvectors.

<u>Corollary C</u>: If a matrix is diagonalised by a similarity transform, i.e. $M' = S^T M S$ is diagonal, and if S is not the same as the matrix of eigenvectors (up to re-ordering the columns) then S is <u>not</u> orthogonal.

(2) Normal Modes Are Not Orthogonal (!)

Normal modes are the vector solutions of,

$$(K - \omega_i^2 M)\overline{\phi}_i = 0 \qquad \text{(Equ.5)}$$

for real symmetric matrices K and M. There is no *a priori* reason to expect the normal modes to be orthogonal, in the above defined sense. Consider the case when the mass matrix is not singular (not always true, but it suffices to establish the exception). In this case we can re-write the dynamic equation, Equ.5, in standard eigenvector form as,

$$M^{-1}K\overline{\phi}_i = \omega_i^2\overline{\phi}_i \qquad \text{(Equ.6)}$$

Now $M^{-1}K$ is in general <u>NOT</u> symmetric, despite both M and K being symmetric. (In fact, even when M is diagonal $M^{-1}K$ will not be symmetric in general). So, there is no reason for the normal modes to be orthogonal in the above defined sense, i.e. we can expect,

$$\overline{\phi}_j^T\overline{\phi}_i \neq 0 \qquad \text{(Equ.7)}$$

Nevertheless it follows simply from Equ.5 that,

$$\overline{\phi}_j^T K\overline{\phi}_i = \overline{\phi}_j^T M\overline{\phi}_i = 0 \qquad (\text{if } \omega_i \neq \omega_j) \qquad \text{(Equ.8)}$$

Standard texts (confusingly!) seem to use Equ.8 as the definition of ‘orthogonal’ in the context of normal modes. See for example “Theory of Vibrations” by Thomson & Dahleh, Equs.6.6.6 and 6.6.7. It would be better to refer to Equ.8 as “weak” or “generalised” orthogonality, or something different anyway.

It follows from Equ.8 that the Modal Matrix, defined by the normal modes,

$$P = \left(\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3...\right) \qquad \text{(Equ.9)}$$

provides a similarity transform which diagonalises both K and M, <u>but P is not orthogonal</u> by virtue of Equ.7.

In fact, P could not possibly be orthogonal, by virtue of Theorem 1. This is because, being a real symmetric matrix, K has orthogonal eigenvectors. Hence K can be diagonalised by an orthogonal transformation composed of these eigenvectors, like O in Equ.4. The same is true of M, but the orthogonal matrix of eigenvectors which diagonalises M, call it O_M, will in general be different from that which diagonalises K, called O_K. But it is clear from Equ.5 that, unless M is a multiple of the unit matrix, the normal modes $\overline{\phi}_i$ cannot be the eigenvectors of K (and vice-versa). Hence P differs from O_K, and, since P diagonalises K, Theorem 1 implies that P cannot be orthogonal.

In summary, K is diagonalised by an orthogonal matrix O_M and also by a non-orthogonal matrix P. The latter is composed of the normal modes, which are not orthogonal. Similarly, M is diagonalised by an orthogonal matrix O_M, which will generally differ from O_K, and also by the same non-orthogonal matrix, P, which diagonalises K.

Example: See “Theory of Vibrations” by Thomson & Dahleh, Example 6.7.1, where the normal modes, and hence P, are clearly not orthogonal.

(3) The Expansion Theorem: Any vector can be expressed as a unique linear combination of the normal modes.

This would be immediately obvious for orthogonal modes because these vectors would then form a mutually perpendicular 'coordinate system'. However, the "weak" orthogonality condition, i.e. Equ.8, is sufficient to guarantee this crucial property, as follows: If an arbitrary vector is written $\bar{v} = \sum_k A_k \bar{\phi}_k$ then the coefficients clearly must be $A_k = \frac{\bar{\phi}_k^T M \bar{v}}{\bar{\phi}_k^T M \bar{\phi}_k}$, noting that the same A_k would result if M were replaced by K in this formula. What is needed is to establish that the normal modes span the whole space, otherwise only vectors within a subspace could be written $\bar{v} = \sum_k A_k \bar{\phi}_k$. This is equivalent to requiring that all the normal modes be linearly independent, which in turn is equivalent to requiring that the determinant of P be non-zero. That this must be the case follows from the similarity transform: $K' = P^T K P$, where K' is diagonal. Since the determinant of the transpose of a matrix equals the determinant of the matrix, it follows that $\|K'\| = \|P\|^2 \|K\|$. But each diagonal element of K' is (twice) the energy associated with that normal mode, i.e. $K'_{ii} = \bar{\phi}_i^T K \bar{\phi}_i$ (no summation). In general, therefore, these terms will be positive definite. Since the determinant of K' is non-zero, it therefore follows that the determinant of P must be non-zero. This establishes that an arbitrary vector can be written $\bar{v} = \sum_k A_k \bar{\phi}_k$.

So, the usual assumption that all possible motions can be expressed as a sum over normal mode motions is still correct (phew!) despite the normal modes not being orthogonal in the conventional sense[1].

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[1] The only problem appears to be if there is a zero energy mode (sometimes called a "mechanism"). This would be a mode of deformation which resulted in no resistance, i.e. no forces would occur under such a displacement. But this does not seem likely in physically realistic structures – unless there really is some free articulation.