Noether's Theorem and the Origin of Spin

Throughout this Section we work in units with c = 1

1. What is Noether's Theorem?

Noether's Theorem states that for every continuous symmetry of a Lagrangian dynamical system there corresponds a conserved quantity.

For example, the absence of an explicit time dependence in the Lagrangian implies that the dynamical behaviour of the system will be the same tomorrow as it is today and was yesterday. This means that the Lagrangian is invariant under the action of translation in time. Hence there must be a conserved quantity. It turns out that this conserved quantity is energy. The conservation of energy for an arbitrary Lagrangian field theory is derived here.

A second example is invariance under spatial translations. The physical laws controlling the behaviour of a system are expected to be the same here as on Mars, or our space programmes will be in big trouble. What is the corresponding conserved quantity? Since there are three spatial dimensions, there are three conserved quantities – the three components of momentum. The conservation of momentum for an arbitrary Lagrangian field theory is derived here.

The third example is invariance under spatial rotations. The isotropy of space implies there should be a corresponding conserved quantity. There is. It is the angular momentum, the conservation of which is also derived below.

All the above examples are for symmetries of the Poincare group. So what about invariance under boosts - the equivalence of inertial observers? What quantity is conserved as a result? There is such a conserved quantity. There must be by Noether's Theorem. However the status of this quantity is rather different – which is why it has no familiar name. We explain this below.

The Poincare group does not provide the only possible symmetries. There may be 'internal' symmetries, independent of spacetime. For example, a Lagrangian for a complex field may depend only upon absolute magnitudes and hence be invariant if the field is multiplied by a phase factor. This leads to a conserved quantity which can be interpreted as electric charge.

In a similar manner the various internal symmetry groups of particle physics [SU(2), SU(3)] lead to a multitude of conserved quantum numbers. Often these quantum numbers (charm, strangeness, etc.) are conserved only under the action of a limited number of the fundamental forces, generally only for very short periods. This is because some of the forces of nature do not respect the corresponding symmetry.

In this article we shall concentrate on the Poincare symmetries as they apply to fields of arbitrary spin. However we shall also indicate how the conserved quantities corresponding to internal symmetries can be derived.

This analysis also shows where spin comes from. The spin degrees of freedom are assumed present in the fields from the start. They correspond to covariance under Lorentz transformations, as discussed elsewhere. However, in this analysis the reason why these degrees of freedom are called "spin" is justified. They are shown to carry angular momentum.

2. The Poincare Symmetries and Their Conserved Quantities

The fields in question will be written $\phi_r(x)$. The subscript r serves two different purposes. Firstly it labels the spin degrees of freedom of a given field. Secondly it can also address the presence of more than one field in the system. The Euler-Lagrange equations for the Lagrangian $L(\{\phi_r\},\{\phi_{r,\mu}\})$ are,

$$\frac{\partial L}{\partial \phi_r} = \partial_\mu \left(\frac{\partial L}{\partial \phi_{r,\mu}} \right) \qquad (1)$$

This applies for all *r*, covering all spin degrees of freedom of all contributing fields.

It is important to distinguish clearly between changes due to coordinate transformations and changes due to transformations amongst the spin degrees of freedom of a field at a point. We shall use x and x' to refer to the *same point in spacetime* as seen by two observers, S and S'. Similarly, the *same spinorial or tensorial field* seen at this point has components ϕ_r as seen by S, but components ϕ_r' as seen by S'. For a scalar field there is therefore no difference, $\phi' = \phi$. However, care is needed because a spacetime transformation can affect both the coordinates of the point in question and the components of a spinorial/tensorial field. The effect of a general, but infinitesimal, Poincare transformation on the coordinates is,

$$x'_\alpha = x_\alpha + \varepsilon_{\alpha\beta} x^\beta + \Delta_\alpha \qquad (2)$$

Recall that x and x' refer to the same point in spacetime, and the transformation (2) describes how to find the coordinates seen by observer S' given those seen by S. However they are both observing the same field at the same point. The $\varepsilon_{\alpha\beta}$ are anti-symmetric and consist of the rotation angles and boost parameters (see §??).

Observer S' sees the field $\phi_r'(x')$ whereas S sees the field $\phi_r(x)$. Though these are the same field at the same point, the differing orientation and/or states of motion of the two observers means that they do not see the same numerical field components. Instead they are related by whatever spin/tensor representation of the Lorentz group is appropriate for that field type. Hence we can write, for an arbitrary but infinitesimal transformation,

Spin > 0: $$\phi_r'(x') = \phi_r(x) + \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta} \phi_s(x) \qquad (3a)$$

Spin = 0: $$\phi_r'(x') = \phi_r(x) \qquad (3b)$$

where repeated indices are summed, in the case of Greek indices over the 4 spacetime dimensions, and for Latin indices over the spin degrees of freedom for the field in question. If there is more than one field, Equ.(3) applies separately for each one (i.e., for different ranges of the subscript *r*). In (3a) the rotation/boost parameters $\varepsilon_{\alpha\beta}$ are, of course, necessarily identical to those appearing in (2), since they are defined by the relationship between S and S'. Hence, since the $\varepsilon_{\alpha\beta}$ are anti-symmetric, the representation matrices of the Lorentz group appearing in (3a) can be assumed anti-symmetric also, i.e., $S_{rs}^{\alpha\beta} = -S_{rs}^{\beta\alpha}$.

I used to get tied in knots when deriving Noether's conserved quantities as a result of the different field differentials that occur. It is important to distinguish between three things,

The local differential: $$\delta_L \phi_r(x) \equiv \phi'_r(x') - \phi_r(x) = \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta} \phi_s(x) \qquad (4a)$$

or, for spin zero (scalar) $$\delta_L \phi_r(x) \equiv \phi'_r(x') - \phi_r(x) = 0 \qquad (4b)$$

The functional differential: $$\delta\phi_r(x) = \phi'_r(x) - \phi_r(x) \qquad (5)$$

The gradient: $$\partial\phi_r(x) \equiv \phi_r(x') - \phi_r(x) = \frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha \qquad (6)$$

The three are related since $\delta\phi_r(x') + \partial\phi_r(x) = \phi'_r(x') - \phi_r(x') + \phi_r(x') - \phi_r(x) = \delta_L \phi_r(x)$. To first order in small quantities this gives,

$$\delta_L \phi_r(x) = \delta\phi_r(x) + \partial\phi_r(x) \qquad (7)$$

since $\delta\phi_r(x)$ only differs from $\delta\phi_r(x')$ by a second order term. Re-arranging and substituting (4) and (6) gives,

$$\delta\phi_r(x) = \delta_L \phi_r(x) - \partial\phi_r(x) = \frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta} \phi_s(x) - \frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha \qquad (8)$$

In (8) it is understood that the term in $S_{rs}^{\alpha\beta}$ is dropped for a scalar field.

Now the two observers must see the same Lagrangian, i.e.,

$$L(\{\phi'_r(x')\}, \{\phi'_{r,\mu}(x')\}) = L(\{\phi_r(x)\}, \{\phi_{r,\mu}(x)\}) \qquad (9)$$

(The alert reader may be worried that because *L* is a Lagrange density, the change in volume element between observers might mean that *L* should not be invariant. This would indeed be the case if L were a density in the sense of "per unit spatial volume". But, in fact, *L* is a density per volume element of spacetime, i.e., per d^4x not per d^3x, and the 4-volume element is a scalar, and therefore so is *L*).

In general, the total change of a function-of-a-function when the independent functions are changed and the point at which they are evaluated is also changed, is given by, for example,

$$df(g(x), h(x), \ldots) = \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial h}\delta h + \ldots + \frac{\partial f}{\partial x}\delta x \qquad (10)$$

Applying this to the difference between the LHS and RHS of (9) gives, (11)

$$dL = L(\{\phi'_r(x')\}, \{\phi'_{r,\mu}(x')\}) - L(\{\phi_r(x)\}, \{\phi_{r,\mu}(x)\}) = \frac{\partial L}{\partial\phi_r}\delta\phi_r + \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = 0$$

On the RHS of (10), evaluation of all quantities at *x* is understood. Here $\delta\phi_r$ corresponds to the differential defined by (5) and given by (8). The Euler-Lagrange equations, (1), allow us to re-write (11) as,

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\right)\delta\phi_r + \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = \partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\delta\phi_r\right) + \frac{\partial L}{\partial x_\alpha}\delta x_\alpha = 0 \qquad (12)$$

Now substituting for $\delta\phi_r$ from (8) gives,

$$\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{1}{2}\varepsilon_{\alpha\beta}S_{rs}^{\alpha\beta}\phi_s(x)-\frac{\partial\phi_r(x)}{\partial x_\alpha}\delta x_\alpha\right]\right)+\frac{\partial L}{\partial x_\alpha}\delta x_\alpha=0 \quad (13)$$

Again it is understood in (13) that the term in $S_{rs}^{\alpha\beta}$ is dropped for a scalar field. This is the master equation from which the conservation laws corresponding to all types of symmetry can be derived.

2.1 The Energy-Momentum Tensor

It is convenient to consider spacetime translations separately from rotations and boosts. In (2), the former have non-zero Δ_α but $\varepsilon_{\alpha\beta}=0$, whereas for the latter it is the other way around. For spacetime translations, $\delta x_\alpha = x'_\alpha - x_\alpha = \Delta_\alpha$, and (13) becomes,

$$\left\{\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right]\right)-\frac{\partial L}{\partial x_\alpha}\right\}\Delta_\alpha=0 \quad (14)$$

But note that $\frac{\partial L}{\partial x_\alpha}\equiv\partial^\alpha L=\eta^{\alpha\mu}\partial_\mu L$, where $\eta^{\alpha\mu}$ is the Minkowski metric tensor. So (14) becomes,

$$\left\{\partial_\mu\left(\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right]-\eta^{\mu\alpha}L\right)\right\}\Delta_\alpha=0 \quad (15)$$

This is true for arbitrary Δ_α and hence the rank 2 tensor,

$$T^{\mu\alpha}\equiv\frac{\partial L}{\partial\phi_{r,\mu}}\left[\frac{\partial\phi_r(x)}{\partial x_\alpha}\right]-\eta^{\alpha\mu}L \quad (16)$$

has zero divergence,

$$\partial_\mu T^{\mu\alpha}=0 \quad (17)$$

Hence the four quantities defined by,

$$P^\alpha=\int T^{0\alpha}\cdot d^3x \quad (18)$$

are conserved, i.e., (17) implies that,

$$\frac{dP^\alpha}{dt}=0 \quad (19)$$

In (18) it is understood that the spatial integral extends over the whole of space. Note, however, that being divergence free, (17), expresses the more general property of continuity. The rate of change of the quantity in question within any region of space is balanced by the flux of the quantity out of the boundary of the region,

$$\frac{\partial}{\partial t}\int_V T^{0\alpha}\cdot d^3x=\int_V\partial_i T^{i\alpha}d^3x=\oiint_{\delta V}T^{i\alpha}dS_i \quad (20)$$

Global conservation, (19), follows from local continuity, (20), in the limit that the region *V* is the whole of space and assuming that the fields vanish at infinity such that

$\oiint_{\delta V} T^{i\alpha} dS_i \to 0$ as the boundary $\delta V \to \infty$.

From these purely mathematical observations we conclude that there are 4 types of 'stuff', labelled by the index α, whose density is $T^{0\alpha}$ and whose vectorial flux through unit area per unit time is $T^{i\alpha}$.

We may suspect that the P^{α} can be interpreted as an energy-momentum 4-vector due to their origin in variations δx^{α} and due to the correspondence in quantum mechanics $\hat{P}^{\alpha} \leftrightarrow i\hbar\partial^{\alpha}$. Actually we do not even need quantum mechanics to make this link since we know that an infinite dimensional unitary representation of the translation operators of the Poincare group is provided by $\hat{P}^{\alpha} \leftrightarrow i\partial^{\alpha}$. Consequently $T^{\mu\alpha}$ is called the energy-momentum tensor. Confidence that this is an appropriate interpretation is gained by consideration of specific cases, including the classical fields. This will be treated in a separate article.

Assuming this interpretation is sound, it follows that T^{00} is energy density, and T^{i0} is the vectorial flux of energy through unit area per unit time. However, T^{0i} is the density of momentum in the direction ^{i}. Finally, T^{ji} is the flux of the ith component of momentum per unit area per unit time in the direction j.

These physical interpretations may not immediately suggest that the energy-momentum tensor is symmetrical. Why should the energy flux (T^{i0}) equal the momentum density (T^{0i})? We shall see in §?? that there are good physical reasons why this should be so. Hence we really need the energy-momentum tensor to be symmetrical. To name but one reason straight away, general relativity requires $G_{\mu\nu} = 8\pi G \cdot T_{\mu\nu}$ and so the symmetry of the Einstein tensor, $G_{\mu\nu}$, implies that the energy-momentum tensor must also be symmetric.

But note that there is nothing about the definition (16) of the energy-momentum tensor to suggest that it must be symmetric. Indeed, we shall find that evaluation of (16) for example Lagrangians shows it not to be symmetric! So how can these two observations be consistent? This is discussed further in §??.

2.2 Rotational Symmetry – The Conservation of Angular Momentum and the Origin of Spin

We now consider a general Lorentz transformation of coordinates but with no translation, i.e., $\delta x_{\alpha} = x'_{\alpha} - x_{\alpha} = \varepsilon_{\alpha\beta} x^{\beta}$. Substitution into (13) gives,

$$\partial_{\mu}\left(\frac{\partial L}{\partial \phi_{r,\mu}}\left[\frac{1}{2}\varepsilon_{\alpha\beta} S_{rs}^{\alpha\beta}\phi_s(x) - \frac{\partial \phi_r(x)}{\partial x_{\alpha}}\varepsilon_{\alpha\beta} x^{\beta}\right]\right) + \frac{\partial L}{\partial x_{\alpha}}\varepsilon_{\alpha\beta} x^{\beta} = 0 \tag{21}$$

Putting $\frac{\partial L}{\partial x_{\alpha}} \equiv \partial^{\alpha} L = \eta^{\alpha\mu}\partial_{\mu} L$ this becomes,

$$\partial_{\mu}\left(\frac{\partial L}{\partial \phi_{r,\mu}}\left[\frac{1}{2} S_{rs}^{\alpha\beta}\phi_s(x) - \frac{\partial \phi_r(x)}{\partial x_{\alpha}} x^{\beta}\right] + \eta^{\alpha\mu} L x^{\beta}\right)\varepsilon_{\alpha\beta} = 0 \tag{22}$$

Note that the term x^β can be taken inside the derivative ∂_μ because $\partial_\mu\left(\eta^{\alpha\mu}x^\beta\varepsilon_{\alpha\beta}\right)=\eta^{\alpha\mu}\varepsilon_{\alpha\beta}\delta_\mu{}^\beta=\eta^{\alpha\beta}\varepsilon_{\alpha\beta}=0$ by virtue of the anti-symmetry of $\varepsilon_{\alpha\beta}$. The second and third terms in (22) can be re-written in terms of the energy-momentum tensor using (16) as,

$$\partial_\mu\left(\frac{1}{2}\cdot\frac{\partial L}{\partial\phi_{r,\mu}}S_{rs}^{\alpha\beta}\phi_s(x)-T^{\mu\alpha}x^\beta\right)\varepsilon_{\alpha\beta}=0 \qquad (23)$$

However we cannot conclude that the coefficient of $\varepsilon_{\alpha\beta}$ in (23) is zero because the anti-symmetry of $\varepsilon_{\alpha\beta}$ means that it is only the part of this which is antisymmetric in $\alpha\beta$ which must be zero. Hence we write,

$$\Im^{\mu\alpha\beta}=\frac{\partial L}{\partial\phi_{r,\mu}}S_{rs}^{\alpha\beta}\phi_s(x)+\left(x^\alpha T^{\mu\beta}-x^\beta T^{\mu\alpha}\right) \qquad (24)$$

Note that $S_{rs}^{\alpha\beta}=-S_{rs}^{\beta\alpha}$, so that $\Im^{\mu\alpha\beta}=-\Im^{\mu\beta\alpha}$, and $\partial_\mu\Im^{\mu\alpha\beta}\varepsilon_{\alpha\beta}=0$ then implies,

$$\partial_\mu\Im^{\mu\alpha\beta}=0 \qquad (25)$$

Consequently we have again found conserved quantities labelled by $\alpha\beta$. Since $\Im^{\mu\alpha\beta}=-\Im^{\mu\beta\alpha}$ there are six conserved quantities defined by,

$$M^{\alpha\beta}=\int\Im^{0\alpha\beta}\cdot d^3x \qquad (26)$$

As before, $\Im^{i\alpha\beta}$, for spatial indices ^{i}, can be interpreted as the flux of these quantities. But what are they?

In this section we will consider only the spatial components, M^{ij}. If we look firstly at the second term in (24) – which would be the only term for a scalar field – we find,

$$M_{orbit}^{ij}=\int d^3x\left(x^iT^{0j}-x^jT^{0i}\right) \qquad (27)$$

But our tentative interpretation of T^{0j} is as the jth component of the density of momentum. So $d^3x\cdot x^iT^{0j}$ is the *ij* component of angular momentum due to the field in a small volume element, i.e., due to the momentum in the *j*-direction multiplied by the distance in the *i*-direction. So if *ijk* is a permutation of 1,2,3, then $d^3x\cdot\left(x^iT^{0j}-x^jT^{0i}\right)$ is the angular momentum in the k-direction due to the fields in this small region. So we can interpret M_{orbit}^{ij} as the angular momentum of a scalar field, or that part of the angular momentum of a field with spin degrees of freedom which does not depend upon the spin degrees of freedom.

Specifically, because M_{orbit}^{ij} arises from products of linear momentum and perpendicular distance, it is the ***orbital*** angular momentum of the field.

The first term in (24) must have an interpretation compatible with the second, otherwise it would make no sense to add them together. This allows us finally to interpret the first term in (24) as the angular momentum which arises from the spin degrees of freedom, rather than from products of linear momentum and perpendicular distance. This justifies the use of the word “spin” for the degrees of freedom associated with the rotational transformations under the representation $S_{rs}^{\alpha\beta}$ of the Lorentz group. The term “spin” encapsulates a form of angular momentum intrinsic to the field, rather than resulting from orbital angular momentum.

$$M_{spin}^{ij} = \int d^3x \cdot \frac{\partial L}{\partial \phi_{r,0}} S_{rs}^{ij} \phi_s(x) \tag{28}$$

The definition of the field conjugate to ϕ_r is,

$$\pi_r = \frac{\partial L}{\partial \phi_{r,0}} \tag{29}$$

So (28) can also be written,

$$M_{spin}^{ij} = \int d^3x \cdot \pi_r S_{rs}^{ij} \phi_s(x) \tag{30}$$

The total angular momentum is thus,

$$M^{ij} = M_{orbit}^{ij} + M_{spin}^{ij} \tag{31}$$

It is only this *total* angular momentum which is conserved as a consequence of (25), i.e.,

$$\frac{dM^{ij}}{dt} = 0 \tag{32}$$

This is the origin of spin. It is required for any theory formulated in Lagrangian terms and for which the fields transform under rotations according to a representation of the Lorentz group other than the trivial one-dimensional representation (i.e., other than scalar fields).

Note that there is nothing quantum-mechanical about this. The fields in question could be purely classical. Admittedly there may be a difficulty interpreting the half-integral spin fields as classical fields due to their essentially complex-valued nature. But this still leaves, at least, the vector and tensor fields, which are certainly perfectly good classical fields – and must possess spin.

So why does spin generally appear to be an overtly quantum mechanical concept? The reason, I suspect, is due to the rotation group being the only compact subgroup of the Poincare group. This means that it is the only subgroup which has finite dimensional unitary representations. Consequently it is only the rotations which correspond to observables with eigenstates which have discrete eigenvalues (j, m). Because the angular momentum states are discrete, the quantum properties are more apparent. For example, a spin half particle can be spin up or spin down with respect to a given measurement direction – but no intermediate value will ever be measured. This is obviously and overtly quantum.

Interpreted classically there is no reason why the spin angular momentum given by (28) should not take a continuum of values, rather than being confined to the eigenvalues of the representation S. This is true even if S is the ½-spinor representation or the vector (spin 1) representation. Thus, the spin angular momentum of the classical electromagnetic field can be evaluated using (28), but it is not (classically) confined to discrete values. Thus it is the “measurement theory” of quantum mechanics which distinguishes the quantum from the classical, not the mere existence of spin *per se*. This holds that measurements can only produce results which are eigenvalues of the operator representing the quantity measured. Where the representation is finite, as for the rotations, these eigenvalues are necessarily a discrete set.

Whilst the same measurement theory also applies to, say, the measurement of energy or linear momentum, these correspond to a non-compact subgroup of the Poincare group. Now in the quantum mechanics of single particles (as opposed to field theory) symmetries must be enacted on quantum states as unitary operators (though anti-unitary operators will also do - Wigner’s theorem). But non-compact Lie groups have no finite dimensional unitary representations. Hence translations in Minkowski spacetime must be implemented in quantum mechanics as infinite dimensional unitary representations, i.e., as differential operators acting on a space of continuous functions. Their eigenvalues then take a continuum of values. Hence, whilst the measurement theory of QM still applies, it is not so obvious since there is no discreteness in the set of possible energy or momentum values.

The translation subgroup can be made compact by confining the particle to a box with suitable boundary conditions. This causes the possible energy and momentum states to become quantised into a discreet set, labelled by quantum numbers *l,m,n* in the manner familiar from solving the Schrodinger equation for this problem. But we see now that the origin of this discreteness is the compactness of the relevant Lie group.

2.3 Is There a Conserved Quantity Corresponding to Covariance under Boosts?

Well, we know there is – we have already derived it in the form of Equ.(25). The boosts correspond to one of $\alpha\beta$ being zero. The conserved quantities are,

$$-M^{j0} = M^{0j} = \int \Im^{00j} \cdot d^3x \qquad (33)$$

But, before we examine (33) more closely, two little problems arise. The first is that, whilst we could easily have guessed the previous conservation laws, i.e., the conservation of energy, momentum and angular momentum, there now appears to be no obvious conserved quantity left. What can it be?

The second problem seems to confirm the first. It comes from considering the commutation properties of the generators of the Poincare group (i.e., the Lie algebra), which are (see §??), (34)

$$[L_j, L_k] = i \in_{jkn} L_n \qquad [K_j, K_k] = i \in_{jkn} L_n \qquad [L_j, K_k] = i \in_{jkn} K_n$$

$$[P_j, L_k] = i \in_{jkn} P_n \qquad [P_j, K_k] = i\delta_{jk} P_0 \qquad [P_\mu, P_\nu] = 0$$

$$[P_0, L_k] = 0 \qquad [P_0, K_k] = -iP_k$$

where $\overline{L}$, $\overline{K}$, P_μ are the generators of rotations, boosts and spacetime translations respectively. Now in quantum mechanics the energy P_0 will be equated with the Hamiltonian, and we will interpret the rate of change of a quantity with the commutator: $i\hbar \frac{dQ}{dt} = [Q, P_0]$. Consequently conserved quantities should be associated with operators which commute with P^0. This works for linear momentum, since $[P_j, P_0] = 0$. It also works for angular momentum since $[P_0, L_k] = 0$. And it trivially works for energy, since obviously $[P_0, P_0] = 0$. However, it would appear that the value taken by the boost operator $\overline{K}$ is not conserved because $[P_0, K_k]$ is *not* zero. So just what ***is*** the meaning of (33) and how is it compatible with $[P_0, K_k] = -iP_k$?

It is quite common in text books for authors to shy away from this question altogether. And even when they do not, they usually only address the meaning of *part* of (33). We shall also offer only a partial explanation here, putting off the rest until §?? We have,

$$M^{0j} = \int d^3x \cdot \left\{ \pi_r S_{rs}^{0j} \phi_s(x) + \left(x^0 T^{0j} - x^j T^{00} \right) \right\} \quad (35)$$

However in this article we shall consider the meaning of the second term alone, i.e., the only term which would occur for a scalar field,

$$M_{scalar}^{0j} = \int d^3x \cdot \left\{ \left(x^0 T^{0j} - x^j T^{00} \right) \right\} \quad (36)$$

But $\int d^3x \cdot x_0 T^{0j} = tP^j$ using (18). Similarly, if we normalise $\int d^3x \cdot x^j T^{00}$ by the total energy, $E = \int d^3x \cdot T^{00}$ we see that,

$$\xi^j = \frac{\int d^3x \cdot x^j T^{00}}{\int d^3x \cdot T^{00}} \quad (37)$$

is the position of the centre of mass of the field system. In this context a better name might be "centre of energy", since that is what (37) literally produces. But, of course, there is no difference in relativity theory (apart from the units, i.e., that factor of c^2). Putting these together we see that the quantity which Noether tells us is conserved is,

$$M_{scalar}^{0j} = tP^j - E\xi^j \quad (38)$$

Now since we already know that energy and momentum are conserved, by (19), the conservation of the quantity (38) gives,

$$\frac{d}{dt} M_{scalar}^{0j} = \frac{d}{dt}\left(tP^j - E\xi^j \right) = P^j - E\frac{d\xi^j}{dt} = 0 \quad (39)$$

This tells us that,

$$P^j = E\frac{d\xi^j}{dt} \quad (40)$$

So, the total momentum equals the total energy times the velocity of the centre of mass-energy (recall we are working in units with $c = 1$). This is the expected result, of course, and corresponds exactly to the relativistic momentum of a single particle ($p = Ev = \gamma mv$) and reduces to the non-relativistic value of momentum for $v << c$ ($p = mv$). However, for an arbitrary system of fields, quite possibly many different fields undergoing complicated interactions, we had no right to assume that this relation would hold. The energy and momentum might be widely distributed over a large region. So Noether's theorem has a non-trivial – indeed a crucial – implication when applied to boosts.

Nevertheless we can ask what is the meaning of the quantity in (38) which is conserved? We note that its value depends upon the coordinate system, both the spatial system and the origin chosen for time. It is this latter feature which renders the quantity rather different from the other conserved quantities. By a suitable choice of the zero datum for time, the quantity in (38) can always be chosen to be zero. And since it is conserved, it is always zero.

So now we see the resolution of the apparent conflict between the implications of the Poincare commutators, (34), under a Hamiltonian interpretation, and Noether's theorem, under a Lagrangian interpretation. The commutators are quite right in indicating that there is no ***physical quantity*** of interest arising from invariance under boosts. The entity resulting from the application of Noether's theorem, (38), can be taken to be identically zero, forever.

But, on the other hand, the application of Noether's theorem to the boosts *does* produce an extremely important ***physical equation***, namely (40), which specifies the total momentum in terms of the total energy and the centre-of-mass velocity. This is important and non-trivial and results from the time derivative of (38) being zero, despite the fact that the quantity in (38) is itself of no interest.

So how does the Hamiltonian / commutator perspective reflect this? The answer is that, although $[P_0, K_k] = -iP_k$ implies that there is no interesting conserved quantity associated directly with $\overline{K}$, nevertheless the fact that $[P_0, P_k] = 0$ implies $[P_0, [P_0, K_k]] = 0$. Since $[P_0, K_k] = -i\hbar \dfrac{dK_k}{dt}$ this means that the object whose conservation is interesting will be $\dfrac{dK_k}{dt}$, rather than $\overline{K}$ itself. This aligns exactly with what has been found from Noether's theorem: it is the time derivative of (38) which has the important physical implication.

3. Examples of Noether Conservation for Other Symmetries – The Conservation of Charge

Consider now the case of purely internal symmetries. That is, transformations which mix the fields $\{\phi_r\}$ but have no affect upon spacetime. We can write,

$$\phi'_r(x') = \phi'_r(x) = \phi_r(x) + \varepsilon_\alpha S^\alpha_{rs}\phi_s(x) \tag{41}$$

Since $\delta x_\alpha = 0$, (13) becomes,

$$\partial_\mu\left(\frac{\partial L}{\partial \phi_{r,\mu}}\varepsilon_\alpha S^\alpha_{rs}\phi_s\right) = 0 \tag{42}$$

And hence the conserved quantities are,

$$\aleph^{\mu\alpha} = \frac{\partial L}{\partial \phi_{r,\mu}} S^\alpha_{rs}\phi_s \tag{43}$$

with $$\partial_\mu \aleph^{\mu\alpha} = 0 \tag{44}$$

Hence there is one conserved quantity for each value of α, corresponding to each continuous parameter in the available symmetry, (41). The simplest example is provided by a complex field whose Lagrangian is invariant under an arbitrary change of phase of the field, i.e.,

$$\phi \to \phi' = e^{i\varepsilon}\phi \approx \phi + i\varepsilon\phi \tag{45a}$$

The Hermetian conjugate field transforms as,

$$\phi^+ \to \phi'^+ = e^{-i\varepsilon}\phi^+ \approx \phi^+ - i\varepsilon\phi^+ \qquad (45b)$$

Note that the field and its conjugate count as separate fields, i.e., the subscript *r* in (43) takes two values, say 1 and 2, for which $S_{11} = i$ and $S_{22} = -i$, and $S_{12} = 0$. Hence (43) gives,

$$\aleph^\mu = \frac{\partial L}{\partial \phi,_\mu} i\phi + \frac{\partial L}{\partial \phi,^+_\mu}(-i)\phi^+ \qquad (46)$$

Suppose we have a Lagrangian whose only term in the derivatives of these fields is $\frac{1}{2}(\partial_\mu \phi)^+ \partial^\mu \phi$. Then (46) gives,

$$\aleph^\mu = i\left[\left(\partial^\mu \phi^+\right)\phi - \left(\partial^\mu \phi\right)\phi^+\right] \qquad (47)$$

This $\aleph^\mu$ turns out to be interpretable as the electrical 4-current density. The Noether conserved quantity is thus electric charge, given by,

$$Q = \int \aleph^0 d^3x = \int i\left[\pi\phi - \pi^+\phi^+\right] d^3x \qquad (48)$$

where we have used the conjugate fields $\pi = \frac{\partial L}{\partial \phi_{,0}} = \phi^+_{,0}$.

Note that invariance under a phase change is effectively a symmetry under group U(1). Consequently we see that the conservation of charge is due to U(1) symmetry.

In the same manner, SU(2) and SU(3) symmetries lead to the conservation of the various flavour numbers / hypercharges of particle physics.

Note again that no quantum mechanical elements have been used here. Whilst the formalism looks very like that of quantum fields, actually there is nothing specifically quantum mechanical about the derivation. Admittedly the use of a complex field would be unusual in classical field theory, but apart from this the conservation of charge as a result of a U(1) symmetry could equally be interpreted as classical physics.

4. Loose Ends

We have left two loose ends to be sorted out. Firstly, in §2.1, we alluded to the requirement for the energy-momentum tensor to be symmetrical, but noted that (16) will generally not produce a symmetrical result. Where does this leave us?

The second issue is that, in discussing the interpretation of the Noether quantity conserved under boosts in §2.3, we confined attention to the case of a scalar field. This leaves unexplained what might be the meaning or significance of the spin dependent term in (35).

Both these questions are taken up in §??