Derivation of the Nodal Forces Equivalent to Uniform Pressure for Quadratic Isoparametric Elements

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The displacement vector $\bar{u}(\bar{r})$ at any point $\bar{r}$ within a single element, E, is linearly related to the displacements of all the nodes of that element via the shape functions, $N_I(\bar{r})$, where I = 1, 2, 3,… represents the nodes of the element, thus,

$$\bar{u}(\bar{r}) = \sum_{I \in E} N_I(\bar{r})\bar{\Delta}_I \qquad (1)$$

where $\bar{\Delta}_I$ is the displacement vector of node I. Suppose nodal forces $\bar{F}_I$ are applied to the nodes of element E. The work done by these forces, on element E alone, if they cause increments of displacement $\delta\bar{\Delta}_I$, is,

$$\text{Work Done by Nodal Forces: } \sum_{I \in E} \bar{F}_I \cdot \delta\bar{\Delta}_I = \sum_{I \in E} F_{Ij}\delta\Delta_{Ij} \qquad (2)$$

Alternatively, if a pressure P is applied to a face of element E which is composed of many small elements of area $d\bar{A}$, then the work done would be,

$$\text{Work Done by Applied Pressure: } \int_A \delta\bar{u} \cdot \left(Pd\bar{A}\right) = \int_A \sum_{I \in E} N_I(\bar{r})\delta\bar{\Delta}_I \cdot \left(Pd\bar{A}\right) \qquad (3)$$

where $\delta\bar{u}$ is the increment of the displacement at any point on the element face in question caused by the applied pressure, and $\delta\bar{\Delta}_I$ is the corresponding increment in the nodal displacements of the element. Equ.(1) has been used to derive the RHS of Equ.(3).

Now if the nodal forces in (2) are to be equivalent to the applied pressure in (3), then (2) and (3) must be equal for arbitrary nodal displacements. Hence we require,

$$\bar{F}_I = \int_A N_I(\bar{r})Pd\bar{A} \qquad \text{that is,} \qquad F_{Ij} = \int_A N_I(\bar{r})PdA_j \qquad (4)$$

In Equ.(4), A is the area of the face, or faces, of the element to which pressure is applied. Equ.(4) applies generally even for curved elements with non-planar faces to which non-uniform pressure is applied.

Now consider a rectangular element with uniform pressure applied to one face perpendicular to the x-axis. Equ.(4) becomes,

$$F_{Ix} = P\int_A N_I(\bar{r})dydz\,; \quad F_{Iy} = F_{Iz} = 0 \qquad (5)$$

For a 2D element, one of the integral variables is simply replaced by the element thickness, t.

2D Quadratic Isoparametric Element

The shape functions, N_I, can be written straight down since they are the unique functions which are not more than quadratic in the coordinates and which are equal to unity at the node, I, in question, but zero at all other nodes. Thus, in 2D we have,

Corner Node 1: $$N_1 = -\frac{1}{4}(1-x)(1-y)(1+x+y) \tag{6}$$

Midside Node 2: $$N_2 = \frac{1}{2}(1-x^2)(1-y) \tag{7}$$

Here we are using the usual isoparametric convention that the corner nodes lie at ± 1. Node 1 lies at (-1, -1), and node 2 at (0, -1).

Let the element edge which consists of nodes 1, 2, 3 have pressure P applied. It's normal is the y-direction, and so Equ.(5) gives,

$$F_{Iy} = Pt\int_{-1}^{1} N_I(\bar{r})\Big|_{y=-1} dx \tag{8}$$

Considering firstly the corner node, the nodal force evaluates to be,

$$\begin{aligned} F_{1y} &= Pt\int_{-1}^{1} N_1(\bar{r})dx = Pt\int_{-1}^{1} -\frac{1}{4}(1-x)(1-y)(1+x+y)\Big|_{y=-1} dx \\ &= Pt\int_{-1}^{1} -\frac{1}{2}(1-x)x dx = \frac{Pt}{3} \end{aligned} \tag{9}$$

Now consider the midside node,

$$\begin{aligned} F_{2y} &= Pt\int_{-1}^{1} N_2(\bar{r})dx = Pt\int_{-1}^{1} \frac{1}{2}(1-x^2)(1-y)\Big|_{y=-1} dx \\ &= Pt\int_{-1}^{1} (1-x^2)dx = \frac{4Pt}{3} \end{aligned} \tag{10}$$

Hence, the corner:midside nodal force ratio is 1:4.

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3D Quadratic Isoparametric Element
The shape functions, N_I, can be written straight down since they are the unique functions which are not more than quadratic in the coordinates and which are equal to unity at the node, I, in question, but zero at all other nodes. Thus, in 3D we have, for the face with z = -1,

Corner Node 1: $$N_1 = -\frac{1}{8}(1-x)(1-y)(1-z)(2+x+y+z) \tag{11}$$

Midside Node 2: $$N_2 = \frac{1}{4}(1-x^2)(1-y)(1-z) \tag{12}$$

Here we are using the usual isoparametric convention that the corner nodes lie at ± 1. Node 1 lies at (-1, -1, -1), and node 2 at (0, -1, -1).

Let the element face z = -1 have pressure P applied. It’s normal is the z-direction, and so Equ.(5) gives,

$$F_{Iz} = Pt\int_{-1}^{1} N_I(\bar{r})\Big|_{z=-1} dxdy \tag{13}$$

Considering firstly the corner node, the nodal force evaluates to be,

$$\begin{aligned} F_{1z} &= Pt\int_{-1}^{1} N_1(\bar{r})dx = Pt\int_{-1}^{1} -\frac{1}{8}(1-x)(1-y)(1-z)(2+x+y+z)\Big|_{z=-1} dxdy \\ &= Pt\int_{-1}^{1} -\frac{1}{4}(1-x)(1-y)(1+x+y)dxdy \\ &= Pt\int_{-1}^{1} -\frac{1}{4}\left(1-3xy-x^2(1+y)-y^2(1+x)\right)dxdy \\ &= Pt\int_{-1}^{1} -\frac{1}{4}\left(1-x^2-y^2\right)dxdy = -\frac{Pt}{4}\left(4-\frac{2}{3}\times 2\times 2\right) = -\frac{Pt}{3} \end{aligned} \tag{14}$$

Now consider the midside node,

$$\begin{aligned} F_{2z} &= Pt\int_{-1}^{1} N_2(\bar{r})dx = Pt\int_{-1}^{1} \frac{1}{4}(1-x^2)(1-y)(1-z)\Big|_{z=-1} dxdy \\ &= Pt\int_{-1}^{1} \frac{1}{2}(1-x^2)(1-y)dxdy \\ &= \frac{PT}{2}\left(\frac{4}{3}\right)\times 2 = \frac{4Pt}{3} \end{aligned} \tag{15}$$

Hence, the corner:midside nodal force ratio is -1:4, the same magnitude as in 2D but the opposite sign.

Remember that the above results are for a single element only. Where a node lies on more than one element, each element will make a contribution to the nodal foirce, which will alter the ratio between the corner and midside nodes.

Consider the edge of a 2D mesh to which pressure is applied. For nodes not at the ends of such a boundary, each corner node will lie on two elements, but midside nodes on only one element. Hence the corner:midside nodal force ratio would then be 1:2.

Consider the boundary of a 3D mesh to which pressure is applied. For nodes not at the edges of such a boundary, each corner node will lie on four elements, but the midside nodes on only two elements. Hence the corner:midside nodal force ratio would then be -1:2.