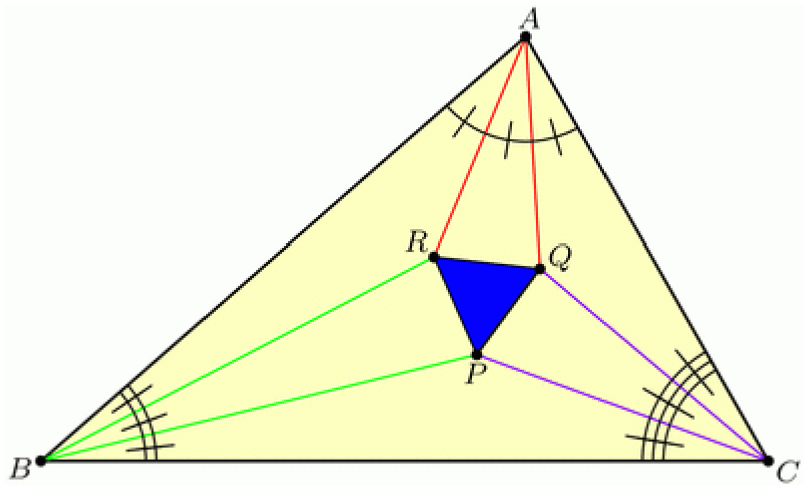
Morley's Theorem

I acknowledge Zachary Abel for Figures 1, 2 and 4, and cut-the-knot.org for Figure 3.

Morley's Theorem is usually expressed thus: "The trisectors of an arbitrary triangle intersect at the vertices of an equilateral triangle".

This is the last of the Great Triangle Theorems. Remarkably it was discovered only in 1899. Strictly, Morley's 1899 work says a great deal more than this – there are, in fact, 27 different equilateral triangles formed from the intersections of trisectors. I shall illustrated all 27 below, but content myself with a proof of the "First Morley Triangle".

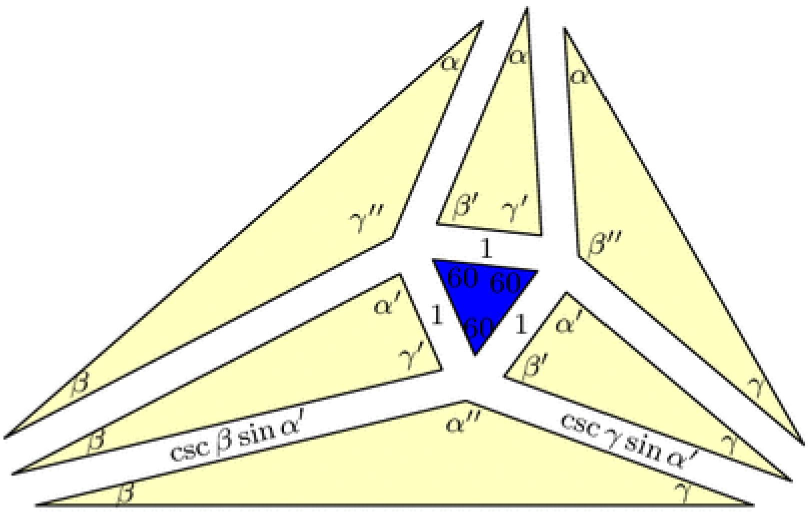
Figure 1: The First Morley Triangle. The theorem says that PQR is equilateral.



The simplest proof is due to John Conway (1995) and works by constructing the original (arbitrary) triangle from seven separate triangles, as illustrated by Figure 2. The angles of the triangle are denoted $3\alpha, 3\beta, 3\gamma$ at the vertices A, B, C respectively, so that $\alpha + \beta + \gamma = 60^o$.

The notation $x' = x + 60^o$ is used throughout. Hence, $x'' = x + 120^o$.

Figure 2: The Conway Construction

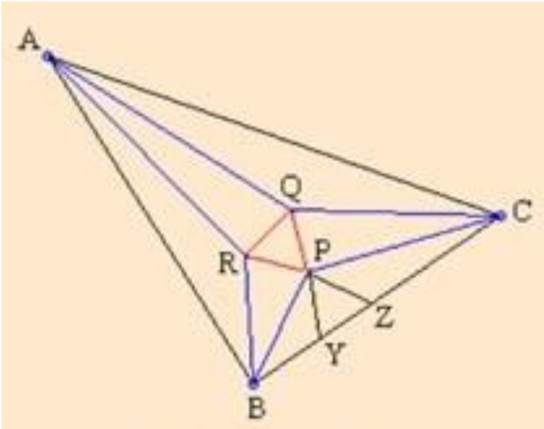


The key to the proof is showing that the seven pieces in Figure 2 really do fit together without gaps or overlaps, and that the central triangle is equilateral. Following Conway we shall show this despite eschewing trigonometry. (The two side lengths indicated in Figure 2 will not be used).

The pieces in Figure 2 are constructed as follows,

[1] Referring to Figure 3, the triangle BPR is defined by the angles β, γ', α'. This is possible because these angles add to $\beta + \gamma' + \alpha' = \beta + \gamma + \alpha + 120^o = 60^0 + 120^o = 180^o$. This defines the shape of BPR. Its size is set by putting RP = 1.

Figure 3: Conway's Further Construction



[2] The triangles CQP and ArQ are defined analogously, with angles γ, α', β' and α, β', γ' respectively and with sides PQ and Qr set to 1. I have written point R as r to indicate that, at this stage, we cannot assume that r = R, i.e. that the three sides of unit length, RP, PQ and Qr form a closed triangle (that is what is to be proved).

[3] The angle BPC is clearly α'' (because then the angles of triangle BPC add to $\alpha'' + \beta + \gamma = \alpha + \beta + \gamma + 120^o = 60^0 + 120^o = 180^o$.

[4] The angles CQA and ARB follow by symmetry (β'' and γ'' respectively).

[5] This sets the shape of triangles BPC, CQA and ARB but not their sizes. Their sizes are specified as follows.

[6] Construct points Y and Z on side BC by setting BPZ to γ' and CPY to β'.

[7] It follows that PZB = PYC = α'so that the angles of triangles BPZ and CPY both add to 180^o.

[8] Hence, triangle PZY is isosceles and we define its sides PZ = PY = 1, which sets the size of triangle BPC.

[9] It follows that triangles BPR and BPZ are congruent (though mirror images) as their angles are equal and the corresponding sides PR and PZ are both 1.

[10] Hence it follows that fixing the sizes of triangles BPR and BPC in the way we have done, above, leads to the same conclusion regarding the length of side BP, i.e., the construction is consistent as regards side BP.

[11] The same construction shows, by symmetry, that the construction is consistent as regards compatibility of the lengths of sides CP, CQ, AQ, AR and BR.

[12] The final step is to note that the construction is also consistent in terms of angular compatibility. This is most easily seen by considering side BP to be rotated around all six

triangles. Rotating anticlockwise by α'' aligns it with PC. Rotating clockwise by γ aligns it with CQ, the total rotation now being $\alpha'' - \gamma$. Continuing in this way will bring BP back into conformance with itself if the total rotation is 360^0, i.e., $\alpha'' - \gamma + \beta'' - \alpha + \gamma'' - \beta = 360^o$, which is indeed an identity.

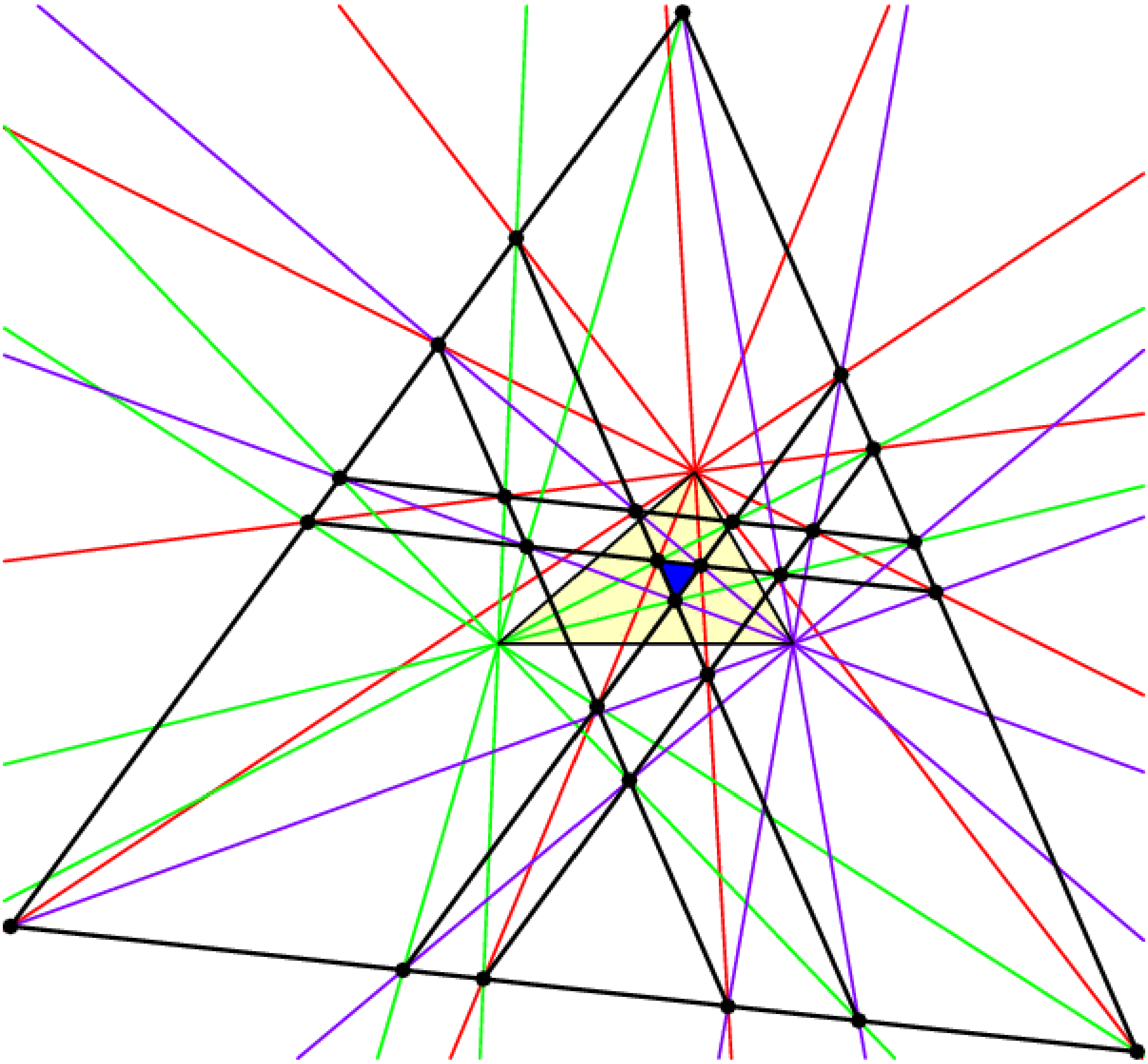
[13] Hence the construction is consistent, PQR is closed and is equilateral by construction.

[14] This may be checked by noting that its angles are all 60^o, e.g., $360^o - (\gamma' + \alpha'' + \beta')$, etc.

QED

BUT there is not just one equilateral triangle, but 27 of them, as illustrated here…

Figure 4: (taken from extraversions | Three-Cornered Things (zacharyabel.com))



This embarrassment of riches results from the use of the trisectors of the ***external*** angles of the triangle, not just the internal angles. Moreover, the external angles can be the supplementary angle (adding to 180^o) or the conjugate angle (adding to 360^o). Hence there are now six trisectors at each vertex. The coloured lines in Figure 4 are the trisectors, whilst the black lines are the equilateral triangles. There are 27 equilateral triangles in Figure 4. You will see that some of the equilateral triangles require trisectors to be extended backwards, increasing the scope yet further. Discounting such liberties, we would be left with 18 equilateral triangles – these are the **18 Morley Triangles**.

In Figure 5, below, I have separated out a random selection of the equilateral triangles to make clearer the trisectors from which they are constructed. In some cases both the supplementary angle trisector and the conjugate angle trisector are required at the same vertex, in which case I distinguish them in terms of colour.

Figure 5