Monstrous Moonshine and the Entropy of the Smallest Black Hole

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Firstly, let me confess that these matters are *way* beyond my competence level. But the hints of a deep connection between the purest of mathematics and the physical world is so fascinating that I can't resist attempting a summary. This brief note concerns two mysteries. The first, the Monstrous Moonshine, lies within pure mathematics and concerns some weird coincidences between certain large integers which occur in two unconnected branches of mathematics. I say "unconnected", but clearly they *are* connected – it's just that no one knew why when this was first observed. The two branches of mathematics concerned are the finite simple groups and the theory of modular functions. The connection has now been elucidated, to a degree, though perhaps not so lucidly as we ordinary mortals might like (Richard Borcherds won the Fields Medal in 1998 for establishing the connection rigorously, see Terry Gannon's review at http://arxiv.org/PS_cache/math/pdf/0402/0402345v2.pdf). Intriguingly, Borcherds employed results from string theory in this work.

The classification of the finite simple groups, a task now complete, was one of the triumphs of 20th century mathematics. But it is a notoriously abstruse area. For a popular level account of the saga I recommend "Symmetry and the Monster" by Mark Ronan. As for the modular functions, the j-function in particular, this goes all the way back to Gauss and Galois, but is also not easy to appreciate at a brief exposure.

The second mystery is the recent suggestion by Ed Witten that these same large integers (or essentially trivial re-castings thereof) can be interpreted as the number of quantum states of a minimal black hole in certain quantum gravity spacetimes.

What's the Monster?

The Monster is the largest sporadic finite simple group. It has this many elements,

808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000

What is a simple group, and what is a sporadic simple group? Firstly the "simple" bit. The idea here is that an arbitrary finite group can, in general, be decomposed into other groups. There is a special type of sub-group, called a normal sub-group, which allows the original group to be considered as made up of this normal subgroup and a 'quotient group'. Thus, the quotient group is written Q = G / N, where G is the original group and N is a normal sub-group. G is effectively decomposed into N and Q. Symbolically, G = N x Q. This process can be repeated on N and Q until we reach the point where the groups have no normal subgroups. In other words, G is expressed (roughly speaking) as a product of factor groups, g1 x g2 x g3 x …, where the factor groups g1, g2,… cannot themselves be factored further. This is analogous to factoring an integer into its prime factors. Thus, the simple groups are just those groups with no normal sub-groups, and which therefore cannot be factored. So the simple groups are analogous to prime numbers.

Now the 'sporadic' bit: there are, of course, an infinite number of different finite groups. Moreover, there are an infinite number of different finite simple groups too, just as there are an infinite number of primes. But almost all fall into a small number of infinite classes. For example, the group of even permutations of n objects (the so-called alternating group, A_n). This is an infinite class of groups since n can take any positive integral value. But after the small number of infinite classes is taken care of, there are still some odd-ball groups left over. These are the 26 sporadic groups. Although some of them share certain family resemblances, basically these 26 sporadic groups are all one-offs. They tend to be very large groups. The Monster is the largest. Bearing in mind the above analogy with the prime numbers, this structure for the finite simple groups is rather surprising. Firstly, we have infinite classes of simple groups of a common type – there is no such feature amongst the primes. Secondly, the sporadic simple groups – which seem more akin in nature to the prime numbers – are finite in number, quite the opposite of what one would expect from the analogy.

As you might guess, finding the sporadic groups and determining their properties was not easy. One of the techniques used in these searches is the "character table". A "representation" is a realisation of the group algebra as a set of square matrices. A representation is reducible if a similarity transformation puts all the matrices in a common direct product form. Otherwise the representation is irreducible. The "character" is the trace of a matrix representation. A "character table" is the set of characters for all the irreducible representations. Since the unit element maps to the unit matrix, its character is just the dimension of the representation. For the Monster, the first few "character degrees", or dimensions of the irreducible representations, are:-

1
196,883
21,296,876
842,609,326
18,538,750,076

What's the Moonshine?

"Moonshine" is what John Conway initially called the suggestion that the above large integers, characteristic of the Monster group, were also to be found in a well known modular function, the j-function. The modular forms play a major role in number theory, thus the connection also links these disparate parts of mathematics. Unfortunately, it is challenging to give a simple account of the j-function. Here's a definition…

The j-function is conventionally defined as $j(\tau) = 12^3 J(\tau)$ where J is Klein's absolute invariant defined by,

$$J(\tau) = \frac{4}{27} \cdot \frac{\left[1 - \lambda(\tau) + \lambda^2(\tau)\right]^3}{\lambda^2(\tau)\left[1 - \lambda(\tau)\right]^2}$$

and where λ is the elliptic lambda function defined in terms of Jacobi theta functions as,

$$\lambda(\tau) = \left[\frac{\vartheta_2(0,q)}{\vartheta_3(0,q)}\right]^4, \quad \text{where, } q = \exp\{i\pi\tau\}$$

In applications, τ is generally the ratio of the half-periods of the elliptic function, and q is known as the *nome*.

The j-function is a modular function. Modular functions are defined as functions of a complex variable which are analytic in the upper half plane except at isolated poles (i.e. meromorphic) and which are invariant under the modular group. The modular group acts on the function by transforming its (complex) variable, z, as a linear fraction,

$$z \to \frac{az+b}{cz+d}$$

The final property required of modular functions is that they are expressible as a Laurent expansion, i.e. there is a smallest negative power z^{-m} in its expansion.

It can be shown that every modular function can be expressed as a rational function of Klein's absolute invariant, J, and that every rational function of J is a modular function. Given this, the j-function is trivially modular. Klein's absolute invariant, J, is so-called because it is invariant under any modular transformation.

The reason for the j-function being of interest is too lengthy to discuss, so suffice it to say that it has played a role in mathematics since Gauss and features strongly in number theory. It also features in the theory of elliptic curves, which provides an alternative, purely algebraic, definition. However, what we are interested in is the Laurent expansion of j in powers of q,

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots,$$

Finally we see the moonshine. The coefficient of 'q' is none other than the dimension of the first non-trivial representation of the Monster group (plus 1). Coincidence? It could have been, but for the fact that the subsequent coefficients of the j-function also have a simple numerical relationship with the dimensions of the Monster group's representations, as follows,

196,884 = 1 + 196,883

21,493,760 = 1 + 196,883 + 21,296,876

864,299,970 = 1 + 1 + 196,883 + 196,883 + 21,296,876 + 842,609,326

Relationships along these lines have been proved to continue for all the expansion coefficients and dimensions. Moreover, this is not all. It turns out that there are further numerical coincidences connecting the j-function and the Monster group.

Where Do Black Holes Come In?

Witten considers a certain quantum gravity theory generated by a conformal field theory in http://arxiv.org/PS_cache/arxiv/pdf/0706/0706.3359v1.pdf. A description is provided by John Baez in http://math.ucr.edu/home/baez/twf_ascii/week254. This stuff is very difficult, and in any case much of Witten's argument does not even claim to be derivation so much as guesswork. The type of spacetime he considers is anti-deSitter spacetime. This has a negative cosmological constant, so the relevance of this suggestion to the real universe is lost on me (since dark energy, or the cosmological constant, appears to be positive in our universe). But Witten is considering a 3d spacetime, as opposed to our 4d spacetime. In 3d spacetime with positive cosmological constant he makes the point that there are no black holes, whereas there *are* black holes if the cosmological constant is negative in 3d.

The conformal theory considered has the interesting property that the cosmological constant is quantised by an integer k = 1, 2, 3… The total vacuum energy of the spacetime is also quantised by this integer. The magnitude of the cosmological constant in our universe is notoriously tiny when expressed in Planck units, of order 10^{-123}. In Witten's 3d spacetime it is $-1/(16k)^2$, and hence, as well as being of different sign, is comparatively enormous in magnitude for modest vales of k (i.e. $\sim 10^{-3}$).

However, for any given k, there is a minimum size of black hole which can exist in this spacetime. What Witten does is to find the number of quantum states of a black hole of minimum size, and how it depends upon k. He does this by arguing that the partition function of the theory should differ from that of the corresponding classical theory only by linear terms, and that it should also be expressible as a power series in the j-function. This leads to a set of functions,

$$Z_1(q) = j(q) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots$$

$$Z_2(q) = j(q)^2 - 393767 = q^{-2} + 1 + 42987520q + 40491909396q2 + \dots$$

$$Z_3(q) = j(q)^3 - 590651j(q) - 64481279$$
$$= q^{-3} + q^{-1} + 1 + 2593096794q + 12756091394048q^2 + \dots$$

$$Z_4(q) = j(q)^4 - 787535j(q)^2 - 8597555039j(q) - 644481279$$
$$= q^{-4} + q^{-2} + q^{-1} + 2 + 81026609428q + 1604671292452452276q^2 + \dots$$

(where j has been redefined for convenience by omitting the constant term, 744).

The coefficient of q in each of the functions Z_k is the number of quantum states of the minimal black hole for that value of k (to an accuracy of within one or two states, at least). Thus, for k = 1, the entropy of the minimal black hole is ln(196884) = 12.190, whereas for k = 4 the entropy is ln(81026609428) = 25.118.

Now the point here is that the entropy of a black hole is also known from semi-classical arguments (i.e. neglecting the quantisation of gravity) by a formula known as the Bekenstein-Hawking formula, which in this case becomes $S = 4\pi\sqrt{k}$. So for k = 1 we expect the result $4\pi = 12.566$, which compares with Witten's 12.190, and for k = 4 we expect $8\pi = 25.133$ which compares with Wittens' 25.118. The comparison for the first 4 values of k is,

k	**Bekenstein-Hawking**	**Witten**	**Difference (%)**
1	12.566	12.190	3.0%
2	17.772	17.576	1.1%
3	21.766	21.676	0.4%
4	25.133	25.118	0.06%

It has been shown that the two become equal asymptotically for large k.

So, it may be that the Monster group has relevance in physics, determining the quantum structure of the most fundamental spacetime features. On the other hand, all this may be just more moonshine.