Chapter ?

The Gold Standard of Unification and the Approach to Relativity

Maxwell's Equations and the Velocity of Light

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Maxwell's equations are,

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \overline{\nabla} \cdot \overline{D} = \rho \qquad (1,2)$$

$$\overline{\nabla} \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J} \qquad \overline{\nabla} \cdot \overline{B} = 0 \qquad (3,4)$$

It is interesting to note how close these four equations are to being just two equations. Equs.(2) and (4) very nearly follow as a consequence of Equs.(1) and (3). This is because $\overline{\nabla} \cdot \left(\overline{\nabla} \times ...\right)$ is identically zero. Hence, taking the ***div*** of Equs.(1) and (3) gives,

$$\frac{\partial}{\partial t} \overline{\nabla} \cdot \overline{B} = 0 \qquad \frac{\partial}{\partial t} \overline{\nabla} \cdot \overline{D} = -\overline{\nabla} \cdot \overline{J} = \frac{\partial \rho}{\partial t} \qquad (5,6)$$

where we have appealed to the conservation of charge, $\overline{\nabla} \cdot \overline{J} = -\frac{\partial \rho}{\partial t}$. Integrating these equations wrt time gives,

$$\overline{\nabla} \cdot \overline{B} = f(\overline{r}) \qquad \overline{\nabla} \cdot \overline{D} = \rho + g(\overline{r}) \qquad (7,8)$$

Here f and g are arbitrary functions of the spatial coordinate $\overline{r}$ but are independent of time (so that they vanish when differentiated wrt time). In non-relativistic physics this is where the matter rests.

However in relativistic physics we smell a rat. Functions of space only cannot be relativistically covariant. A Lorentz boost will cause these space-only functions to take on a time dependence in another frame. But this would undermine the Maxwell equations in this frame. Taking it on trust for a moment that Maxwell's equations turn out to be Lorentz covariant, this implies that the functions f and g must be identically zero. So Equs.(7,8) are in fact identical to Equs.(2,4), and have been deduced from Equs.(1,3).

So we anticipate that the relativistic perspective will cause Maxwell's equations to reduce in number. We will see below that this is correct, but the above observation is helpful in understanding just why this happens.

In vacuum we have,

$$\overline{D} = \varepsilon_0 \overline{E} \qquad \text{and} \qquad \overline{B} = \mu_0 \overline{H} \qquad (9,10)$$

Hence in vacuum and in a region with no sources, Maxwell's equations (1,3) become,

$$\overline{\nabla} \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t} \qquad \overline{\nabla} \times \overline{H} = \varepsilon_0 \frac{\partial \overline{E}}{\partial t} \qquad (11,12)$$

Taking the ***curl*** of (11) and substitution of (12) and use of the identity $\nabla \times \left(\overline{\nabla} \times \overline{E}\right) \equiv \overline{\nabla}\left(\overline{\nabla} \cdot \overline{E}\right) - \nabla^2 \overline{E}$, together with $\overline{\nabla} \cdot \overline{E} = 0$ gives,

$$\nabla^2 \overline{E} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \overline{E} \tag{13}$$

We recognise (13) as the wave equation for a wave propagating at speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

It is worth pausing to be suitably reverential at this point. The great theme of physics is unification. To understand as much as possible from as few principles as possible is our perpetual goal. Surely Maxwell's achievement in identifying the electromagnetic nature of light must be the unification *par excellance*. Yesterday electromagnetism and light were two entirely separate phenomena, yet today they are the same thing. Not forgetting that it was also chiefly Maxwell who was responsible for unifying electricity and magnetism into a single electromagnetic field hence making the identification of the nature of light possible.

The constants ε_0 and μ_0 which appear in the expression for *c* are those which determine the forces between charges and current elements (the Coulomb and Biot-Savart laws). The magnitudes of these constants can be found from experiments using purely electrical equipment, with no optics (light) involved – and no other sort of radiation either for that matter. Yet these constants, apparently unconnected with light, determine the speed of light through the simple expression $c = 1/\sqrt{\mu_0 \varepsilon_0}$. With the values of ε_0 and μ_0 available[1] in 1862, when Maxwell first made this discovery, he deduced the speed of light to be within ~3% of the currently accepted value. As Maxwell himself put it, "*we can scarcely avoid the conclusion that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena*". It is also worth emphasizing that this was crucially a triumph of theory. The electromagnetic nature of light was not an experimental discovery. It was discovered because the complete description of the electromagnetic field through Maxwell's equations permitted a wave-like solution via the simple manipulation given above, and the predicted speed matched that of light.

Leaping forward 43 years we reach Einstein's *annus mirabilis*, 1905. How can Maxwell's equations be consistent with the claim that the velocity of light is the same in all inertial frames? Moreover, if all inertial observers are equivalent, the *form* of Maxwell's equations should look like Equs.(1-4) wrt any observer moving at a constant velocity. So, in a vacuum and with no sources, we require in the frame of observer S',

$$\overline{\nabla}' \times \overline{E}' = -\frac{\partial \overline{B}'}{\partial t'} \qquad \overline{\nabla}' \cdot \overline{E}' = 0 \tag{14,15}$$

$$\overline{\nabla}' \times \overline{B}' = \frac{1}{c^2} \frac{\partial \overline{E}'}{\partial t'} \qquad \overline{\nabla}' \cdot \overline{B}' = 0 \tag{16,17}$$

In (14-17) $\overline{E}'$ and $\overline{H}'$ are the electric and magnetic fields measured by S' whose coordinate system, $(t', \overline{r}')$, is related to the coordinate system of observer S, i.e., $(t, \overline{r})$, by a Lorentz transformation. The operator $\overline{\nabla}'$ in (14-17) is wrt the coordinate system of S', i.e., $\overline{\nabla}' = \hat{x}' \frac{\partial}{\partial x'} + \hat{y}' \frac{\partial}{\partial y'} + \hat{z}' \frac{\partial}{\partial z'}$. Provided that the constants ε_0 and μ_0 are

[1] Strictly their contemporary equivalents

universal to all inertia observers, they will all, by virtue of (14-17) measure the same speed of light, i.e., the same value of $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

Here, then, is the explanation for the result of the Michelson-Morley experiment. The speed of light is the same irrespective of the speed of the source/detector because the electric and magnetic fields seen in the moving frame are transformed from their stationary values in just the manner required to preserve the form, (14-17), of Maxwell's equations.

So far this is consistent with Einstein's contention. But this is less than impressive as yet since we do not know what $\overline{E}'$ and $\overline{H}'$ make (14-17) true, or whether these transformed fields make sense. Before looking at this, note the form of the Lorentz force,

$$\overline{F} = q\left(\overline{E} + \overline{v} \times \overline{B}\right) \tag{18}$$

This says that the force acting on a charge q depends only upon the electric field if the charge is stationary (velocity $\overline{v}$ is zero). However, if the charge is moving it acquires an extra force which depends upon the magnetic field. But if we transform to the frame instantaneously co-moving with the charge, and if the form of physical laws is the same for all inertial observers, the force will again depend only on the electric field as seen in the new frame, i.e., $\overline{F}' = q\overline{E}'$. Now so long as we confine ourselves to speeds small compared with that of light, the forces seen by the two observers must be the same since they are both equal to the mass of the particle times its acceleration, and the acceleration is the same for all observers in uniform motion (in the limit $v << c$). So this implies that the electric field in the moving frame is,

$$\text{For } v << c: \qquad \overline{E}' = \overline{E} + \overline{v} \times \overline{B} \tag{19}$$

Now this is a remarkable way of looking at things because it says that the electric and magnetic fields are not different things at all, but merely different perspectives on the same thing (the electromagnetic field) when seen by observers in different states of motion.

Returning now to (14-17), these also specify how the fields in frame S' are related to those in frame S. This is because we know how coordinates $(t', \overline{r}')$ are related to coordinates $(t, \overline{r})$ under a Lorentz transformation, and (14-17) must reduce to the same equations in frame S with all the dashes omitted. Only if this also implies (19) is Einstein's idea consistent. Consider the case that the Lorentz transform is a boost in the x-direction. The derivatives are transformed as,

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right); \qquad \frac{\partial}{\partial t'} = \gamma\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right); \qquad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}; \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \tag{20}$$

Equs.(14-17) then imply that the transformed fields are,

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - vB_z\right) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E'_z &= \gamma\left(E_z + vB_y\right) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned} \tag{21}$$

Substitution of (20) and (21) into (14-17) shows that they reduce to the same field equations in the frame S, i.e., to,

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \overline{\nabla} \cdot \overline{E} = 0 \qquad (22,23)$$

$$\overline{\nabla} \times \overline{B} = \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} \qquad \overline{\nabla} \cdot \overline{B} = 0 \qquad (24,25)$$

It is educational to go through the algebraic exercise of showing this – if only so you appreciate how much simpler is the explicitly covariant formalism.

Note that Equs.(21) are consistent with (19) in the non-relativistic limit when $\gamma \approx 1$, as required.

So Einstein's programme makes sense. The principle of the equivalence of inertial observers is consistent with the speed of light being the same in all inertial frames, both from the kinematic point of view (transforming the spacetime coordinates) and also from the dynamic perspective when light is understood as an electromagnetic wave. It also makes sense in that the transformation of the electromagnetic field components which is required for Maxwell's equations to be formally the same for all observers is consistent with the motion of charged bodies under the action of the electromagnetic field.

The impact of relativity on electromagnetism can be seen as the completion of Maxwell's programme of unification. Whilst Maxwell showed that the electric and magnetic fields were dynamically inter-related through his equations, Einstein showed that their unity was essentially kinematic. The observer's state of motion will morph electric and magnetic fields into each other. Whilst Maxwell identified the electromagnetic nature of light and what determined its velocity, Einstein showed that the democracy of observers also required this speed of light to be a universal constant.

Exercise: Prove that (14-17) with (20,21) imply (22-25)

Substituting (20) and (21) into Equ.(14) gives,

$$\left(\frac{\partial}{\partial y}\gamma\left(E_z + vB_y\right) - \frac{\partial}{\partial z}\gamma\left(E_y - vB_z\right), \frac{\partial E_x}{\partial z} - \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\gamma\left(E_z + vB_y\right), \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\gamma\left(E_y - vB_z\right) - \frac{\partial E_x}{\partial y}\right) =$$
$$-\gamma\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(B_x, \gamma\left(B_y + \frac{v}{c^2}E_z\right), \gamma\left(B_z - \frac{v}{c^2}E_y\right)\right)$$

This reduces to: $$\left(v\gamma\overline{\nabla} \cdot \overline{B}\right)\hat{x} + \overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad (A)$$

Substituting (20) and (21) into Equ.(17) gives,

$$\gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)B_x + \gamma\frac{\partial}{\partial y}\left(B_y + \frac{v}{c^2}E_z\right) + \gamma\frac{\partial}{\partial z}\left(B_z - \frac{v}{c^2}E_y\right) = 0$$

Which reduces to: $$\gamma\overline{\nabla} \cdot \overline{B} + \gamma\frac{v}{c^2}\left(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = 0 \qquad (B)$$

(A) and (B) together give $\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$ and $\overline{\nabla} \cdot \overline{B} = 0$, in the limit $v/c \sim 0$, i.e., Equs. (22) and (25). In the same way, Equs.(15) and (16) produce Equs.(23) and (24).