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Citation: *Journal of Mathematical Physics* **56**, 102103 (2015); doi: 10.1063/1.4933322

View online: <http://dx.doi.org/10.1063/1.4933322>

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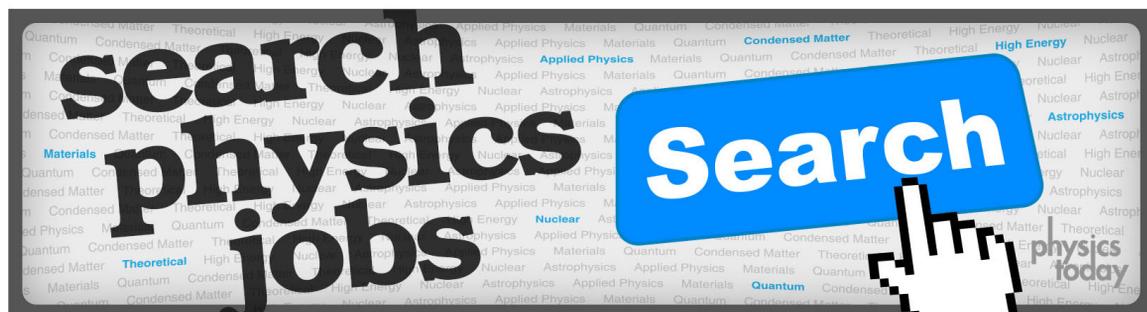
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A spherically symmetric bound state of the coupled Maxwell-Dirac equations with self-interaction alone

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(Received 5 December 2014; accepted 5 October 2015; published online 23 October 2015)

Stationary, static, spherically symmetric solutions of the Maxwell-Dirac system, treated as classical fields, have been found which are localised and normalisable. The solutions apply to any bound energy eigenvalue in the range $0 < E < m$, where m is the bare mass in the Dirac equation. A point charge of any magnitude and either sign may be placed at the origin and the solutions remain well behaved and bound. However, no such central charge is necessary to result in a bound solution. As found previously by Radford, the magnetic flux density is equal to that of a monopole at the origin. However, no monopole is present, the magnetic flux being a result of the dipole moment distribution of the Dirac field. The Dirac field magnetic dipole moment is aligned with the magnetic flux density and so the resulting magnetic self-energy is negative. It is this which results in the states being bound ($E < m$). The case which omits any central point charge is therefore a self-sustaining bound state solution of the Maxwell-Dirac system which is localised, normalisable, and requires no arbitrarily added “external” features (i.e., it is a soliton). As far as the author is aware, this is the first time that such an exact solution with a positive energy eigenvalue has been reported. However, the solution is not unique since the energy eigenvalue is arbitrary within the range $0 < E < m$. The stability of the solution has not been addressed. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4933322>]

I. INTRODUCTION

For a specified electromagnetic field, the Dirac equation determines the Dirac field, subject to suitable boundary conditions. On the other hand, the current associated with a charged Dirac field also acts as a source of the electromagnetic field. Consequently, even if no electromagnetic field is externally imposed, one will arise in the presence of a charged Dirac field. An isolated Dirac field is therefore required to be consistent with its own electromagnetic field. The resulting coupled problem is non-linear. The traditional approach to the coupled Maxwell-Dirac system is via quantum field theory, which effectively linearises the problem via a perturbation series approach. However, there remains a perennial interest in the Maxwell-Dirac equations as a classical system of commuting fields with a real electromagnetic 4-potential. The question which arises is whether this system of equations can shed light on the stability of fundamental charged particles against being blown apart by their own Coulomb repulsion, a question which quantum electrodynamics does not address but avoids. That localised, finite solutions to the coupled Maxwell-Dirac equations exist has been established, under suitable conditions, e.g., Refs. 1–11. In particular, Refs. 6 and 10 demonstrate the existence of stationary solutions of the coupled Maxwell-Dirac equations, albeit with negative energy.

No non-trivial localised, finite (normalisable) solutions to the coupled Maxwell-Dirac equations in closed form are known. A closed form solution which is not localised or finite has been given by Legg, Ref. 12. Several papers claim to have provided localised, finite solutions numerically, notably Wakano, Ref. 13, Lisi, Ref. 14, Bohun and Cooperstock, Ref. 15, and most significantly, Radford, Ref. 16, and Radford and Booth, Ref. 17. In Refs. 13–15 the approximation was made that the

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magnetic vector potential can be neglected and the problem assumed dominated by the electric scalar potential. However, as noted by Radford, Ref. 16, it is fundamentally in conflict with the Dirac equation to assume that the magnetic vector potential is negligible. Consequently, these solutions are questionable even as approximations. Lisi, Ref. 14, does go on to discuss the case in which the magnetic field is included, but for the physical value of the electromagnetic coupling constant his solution has negative energy, approaching minus the field mass, and hence its meaning is obscure. One may wonder what causes the solutions in Refs. 13–15 to be bound given the repulsive nature of the electrostatic self-force. Lisi, Ref. 14, opines that it arises due to the Klein paradox effect. This intuition has been confirmed mathematically in Ref. 10, i.e., that the negative energy bound states of Refs. 10 and 14 arise due to the Klein paradox effect.

Radford, Ref. 16 was the first to offer a localised, finite solution with no approximations, albeit partly numerical. His analysis was based on the assumption of spherical symmetry together with being both stationary and static. The present work revisits this spherically symmetric, stationary, static case. This is motivated by the present author's belief that Radford's solution, Ref. 16, applies for a negative energy eigenvalue of the Dirac field, specifically $E = -m$, and hence, like the Lisi solutions, Ref. 14, its physical meaning is unclear. The present work seeks positive energy eigenvalue solutions.

Moreover, the solution of Ref. 16 is electrically neutral overall, with a point charge at the origin equal and opposite to the Dirac field charge (both of which are far larger in magnitude than the quantum of charge). This point charge at the origin had to be assumed as "externally supplied," and hence undermines the autonomy of the solution. In contrast, the solutions offered here are normalised to unity, so the Dirac field has charge equal to the term q appearing in the Dirac equation. Moreover, we find bound solutions with arbitrary total charge if point charges may be externally supplied at the origin. Most importantly, there is a bound solution for *no* point charge at the origin so that the total charge is just q .

In common with Radford, Ref. 16, we find that the magnetic flux density is equal to that of a monopole. However, we argue that this does not require that a monopole exists or is placed at the origin by an external agency. We interpret the radial magnetic flux density, \vec{B} , with strength proportional to $1/r^2$, as arising simply from the Dirac field spin. Hence, our solution requires neither a point charge nor a monopole at the origin. We offer spherically symmetric, stationary, static, localised, bound solutions to the coupled Maxwell-Dirac equations with positive energy eigenvalues, with no approximations, which are fully self-consistent and without the need for any externally supplied features, albeit the solutions are numerical. As far as the author is aware, this is the first time that such an exact solution with positive energy eigenvalue has been reported. The solutions apply for an arbitrary bound state energy eigenvalue in the range $0 < E < m$ and hence do not define a unique physical mass for the soliton.

Unlike previous solutions, we offer a physically appealing reason for the bound nature of our solution, namely, that it arises from the magnetic attraction between the distributed dipole moments resulting from the Dirac field's spin. However, the solution is not a candidate as a model for the leptons since, being spherically symmetric, it has zero net spin and zero net magnetic moment. Moreover, the stability of the solution has not been established.

II. THE DIRAC AND MAXWELL EQUATIONS

Throughout, we work in units with $\hbar = c = 1$. The Dirac equation is

$$(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)\psi = 0, \quad (2.1)$$

where ψ is the Dirac bispinor field and A_μ is the electromagnetic 4-potential, taken here to be real. Both fields are assumed to be classical, commuting fields, albeit the Dirac field is complex valued. The usual convention is adopted in which Greek indices run over spacetime components whilst Latin indices run over space components only. We work in the time-like convention in which $a^\mu b_\mu = a_0 b_0 - \vec{a} \cdot \vec{b}$. For the sake of keeping track of the correct signs, note that $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and the contravariant x^μ components are identified with the usual coordinates, i.e., $\{x^\mu\} \equiv (t, \vec{r})$. The same applies for the 4-potential, $\{A^\mu\} \equiv (V, \vec{A})$, where V is the electrostatic scalar potential and \vec{A} the magnetic vector potential. We

labour these points to clarify that, in (2.1), we have $\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\nabla}$ whereas $\gamma^\mu A_\mu = \gamma^0 V - \vec{\gamma} \cdot \vec{A}$. We work in the Pauli-Dirac representation of the γ^μ matrices, thus,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.2}$$

where, in Cartesian coordinates we have Pauli matrices,

$$\{\sigma^j\} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \tag{2.3}$$

and in spherical polars these become,

$$(\sigma^\theta, \sigma^\phi, \sigma^r) = \left[\begin{pmatrix} -s & ce^{-i\phi} \\ ce^{i\phi} & s \end{pmatrix}, \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}, \begin{pmatrix} c & se^{-i\phi} \\ se^{i\phi} & -c \end{pmatrix} \right], \tag{2.4}$$

where $c = \cos \theta$, $s = \sin \theta$, so that $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the Minkowski metric. The familiar commutation properties of the Pauli matrices,

$$[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k, \tag{2.5}$$

where ijk is a positive cyclic permutation of the spatial indices, continues to hold for the matrices expressed in spherical polars, (2.4), taking (θ, ϕ, r) as the cyclic order. Defining $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ its spatial components provide the spin operator,

$$\hat{S}^k = \frac{1}{2}\sigma^{ij} = \frac{1}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{2.6}$$

where ijk is again a positive cyclic permutation of the spatial indices. The time components provide the velocity operator, $\alpha^j = i\sigma^{j0} = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$.

In (2.1), q and m are the bare charge and mass of the Dirac field quantum. Taking e to be the quantum of charge, defined as a positive quantity, if (2.1) is to apply to the electron field then $q = -e$. However, we will generally illustrate the results for $q > 0$.

The Dirac current is

$$J^\mu = q\bar{\psi}\gamma^\mu\psi, \tag{2.7}$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$, in terms of which the inhomogeneous Maxwell equations can be written as

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu, \tag{2.8}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.9}$$

and the usual electric field and magnetic flux density can be identified as $E^j = F^{j0}$ and $B^k = -\frac{1}{2}\epsilon_{ijk}F^{ij}$ in terms of which the inhomogeneous Maxwell equations are recovered in the form $\vec{\nabla} \cdot \vec{E} = 4\pi J^0$ and $\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$, where $\vec{E} = -\vec{\nabla}V - \partial_t \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The homogeneous Maxwell equations are identities when expressed in terms of A^μ and need not be considered further. Because signs will be crucial in what follows, we emphasise the elementary observation that a lone positive point charge, q_0 , at the origin results in a positive potential, q_0/r , and an outward pointing electric field $q_0\hat{r}/r^2$.

For later use, we note that if the current happens to be expressible as a tensor divergence,

$$J^\nu = \partial_\mu \Omega^{\mu\nu} \tag{2.10}$$

then integrating (2.8) gives simply

$$F^{\mu\nu} = 4\pi\Omega^{\mu\nu} + \Lambda^{\mu\nu}, \tag{2.11}$$

where $\Lambda^{\mu\nu}$ is some anti-symmetric tensor with zero divergence, $\partial_\mu \Lambda^{\mu\nu} = 0$, and we may take $\Lambda^{\mu\nu}$ to be zero if the absence of any sources can be assumed to imply no electromagnetic field.

III. THE GORDON DECOMPOSITION

It was appreciated almost immediately after Dirac published his equation that the Dirac current, (2.7), involves more than just the electric current due to the motion of charge, i.e., more than just the “convection current.” The Dirac equation, (2.1), implies the following decomposition of the Dirac current, as first noted by Gordon, Ref. 18:

$$J^\nu = q\bar{\psi}\gamma^\nu\psi = J_{conv}^\nu + J_{spin}^\nu = \frac{q}{2m}\bar{\psi}\left(\hat{\pi}^\nu + \hat{\pi}^{\nu\leftarrow}\right)\psi + \frac{q}{2m}\partial_\mu(\bar{\psi}\sigma^{\nu\mu}\psi), \quad (3.1)$$

where $\hat{\pi}^\mu = i\partial^\mu - qA^\mu$ is the kinetic momentum operator. Note that both the convection current, J_{conv}^ν , and the spin current, J_{spin}^ν , are real. Since the spin current is of the form of a tensor divergence, the electromagnetic field generated by the spin current is, from (2.10) and (2.11), simply

$$F_{spin}^{\mu\nu} = \frac{4\pi q}{2m}(\bar{\psi}\sigma^{\nu\mu}\psi). \quad (3.2)$$

In particular, the spatial components lead to the following magnetic flux density in terms of the spin operator matrix element:

$$\bar{B}_{spin} = \frac{4\pi q}{m}\bar{\psi}\bar{S}\psi \equiv 4\pi\bar{M} \quad (3.3)$$

noting that this is only part of the total magnetic field in the general case. The RHS of (3.3) recognises that q/m is just two Bohr magnetons, which, when multiplied by the spin operator matrix element for a particle with gyromagnetic ratio 2 gives the magnetic moment per unit volume, \bar{M} . For a magnetic medium, (3.3) would be the relationship between the magnetic flux density and the magnetisation in the case that the magnetic field strength, \bar{H} , were zero.

Whilst an elementary observation, (3.3) serves to remind that the Dirac field is not just a distribution of charge but also a distribution of magnetic moment, which will generally produce a magnetic flux density, \bar{B} , even if the convection current is zero. Approximations which neglect the \bar{B} field are therefore inherently contradictory. Eqs. (3.1)–(3.3) are not required in what follows. These remarks have been made simply to emphasise the importance of the electron spin as a source of magnetic flux.

IV. INVERSION OF THE DIRAC EQUATION

In the case that the electromagnetic 4-potential is real, and assuming that $\bar{\psi}\psi$ is non-zero, the Dirac equation, (2.1), can be inverted by simple algebraic manipulation to provide an expression for the 4-potential in terms of the Dirac field (see, for example, Ref. 19),

$$A^\nu = \frac{i\left[\bar{\psi}\gamma^\nu\partial\psi - \bar{\psi}\overleftarrow{\partial}\gamma^\nu\psi\right] - 2m\bar{\psi}\gamma^\nu\psi}{2q\bar{\psi}\psi} = -\frac{\text{Im}(\bar{\psi}\gamma^\nu\partial\psi) + mJ^\nu/q}{q\bar{\psi}\psi}. \quad (4.1)$$

This inversion fails if $\bar{\psi}\psi = 0$. It will be seen that the class of solutions developed here have $\bar{\psi}\psi \neq 0$ consistent with the inversion of Eq. (4.1).

V. STATIONARY, STATIC, SPHERICALLY SYMMETRIC FIELDS

A stationary state is an energy eigenvalue, so that its time dependence may be factored out as $\psi(x) = \psi(\bar{r})e^{-iEt}$. From here on we shall assume an energy eigenstate and by ψ we shall understand $\psi(\bar{r})$.

A static state has zero 3-current, $J^k = 0$.

It is simplest to adopt a definition of “spherically symmetric” which requires that the Dirac field has its spin oriented radially everywhere. The remaining requirements for spherical symmetry are that the Dirac field probability density and the electric field and magnetic flux density are functions of r only, and that the latter are oriented radially. [We note in passing that it can be shown that the radial orientation of the Dirac spins is implied by the other parts of the definition of spherical symmetry.]

Radially oriented spins mean that the Dirac field must be an eigenfunction of the radial spin operator given by (2.6) where σ^r is given by (2.4). This requires that the Dirac field be of the form

$$\psi = e^{i\kappa} \begin{pmatrix} u \\ i\lambda e^{i\xi} u \end{pmatrix}, \tag{5.1}$$

where

$$u = u_{in} = F \begin{pmatrix} -e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} \end{pmatrix} \tag{5.2}$$

or

$$u = u_{out} = F \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} \end{pmatrix}, \tag{5.3}$$

where F , λ , κ , and ξ are real functions, potentially of all of the spherical polar coordinates r, θ, ϕ . We may set the function κ to zero by the gauge transformation $\psi \rightarrow \psi e^{-i\kappa}$ which, from (4.1), results in $2qA^\nu \rightarrow 2qA^\nu - \partial^\nu \kappa$ and which leaves the electromagnetic field, (2.9), invariant.

The two spinors u_{in} and u_{out} have spins pointed radially inwards and radially outwards, respectively, i.e.,

$$\hat{S}^r u_{in} = -\frac{1}{2} u_{in}, \quad \hat{S}^r u_{out} = \frac{1}{2} u_{out}. \tag{5.4}$$

The requirement that the solution be static ($J^k = 0$) implies that the function ξ must be zero, as follows:

$$\begin{aligned} J^k &= \bar{\psi} \gamma^k \psi = (u^\dagger (i\lambda e^{i\xi} u)^\dagger) \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \begin{pmatrix} u \\ i\lambda e^{i\xi} u \end{pmatrix} \\ &= u^\dagger \sigma_k (i\lambda e^{i\xi} u) + (i\lambda e^{i\xi} u)^\dagger \sigma_k u \\ &= -2\lambda u^\dagger \sigma_k u \cdot \sin \xi = 0. \end{aligned} \tag{5.5}$$

Hence, $\xi = 0$ and we can replace (5.1) with

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ i\lambda u \end{pmatrix}. \tag{5.6}$$

From (5.2) to (5.6), it follows that $J^0 = q\psi^\dagger \psi = (1 + \lambda^2) F^2$ which is required to depend upon r only. A sufficient condition is that F and λ each depend upon r alone. In fact, this is also necessary in order that the substitution of (5.2) and (5.6) into (4.1) for $\nu = t$ produces a spherically symmetric potential. In passing we note that the potential from (4.1) is then

$$V = \frac{E}{q} + \frac{\partial_r \lambda + 2\lambda \frac{\partial_r F}{F} + \frac{2\lambda}{r} - m(1 + \lambda^2)}{q(1 - \lambda^2)}, \tag{5.7}$$

where E is the energy eigenvalue, and we are assuming $\lambda^2 \neq 1$. With the form of Dirac field given by (5.2) and (5.6) and with F and λ dependent upon r only, not only is the Dirac 3-current zero, $J^k = 0$, but its convection and spin parts are separately zero, $J^k_{conv} = J^k_{spin} = 0$. The vanishing of the spin current is consistent with Eq. (3.3).

VI. THE MONOPOLE FIELD

Substituting (5.6) into (4.1) allows the 4-potential to be found in terms of the Dirac field. To do this recall that in spherical polars,

$$\bar{\sigma} \cdot \bar{\nabla} = \sigma^r \partial_r + \frac{\sigma^\theta}{r} \partial_\theta + \frac{\sigma^\phi}{r \sin \theta} \partial_\phi. \tag{6.1}$$

For definiteness we use the inward pointing spinor, (5.2). In the first term in the numerator of (4.1) for $\nu = r$ or $\nu = \theta$, $\bar{\psi} \gamma^\nu \not{\partial} \psi$ is purely real and hence the first term is zero. Since $J^k = 0$, we find $A^r = A^\theta = 0$. However, for $\nu = \phi$ the numerator in (4.1) evaluates to $\frac{(1-\lambda^2)}{r} F^2 \cot \theta$ and the denominator is $2q(1-\lambda^2)F^2$ so we find

$$A^\phi = \frac{\cot \theta}{2qr}. \tag{6.2}$$

From which we get,

$$\bar{B} = -\frac{\hat{r}}{2qr^2}. \tag{6.3}$$

This confirms the magnetic monopole field as found by Radford, Ref. 16. However, this does not imply that an externally supplied point monopole is present at the origin. In fact, it is rather a fluke that (6.3) obeys the free space Maxwell equation (for $r \neq 0$) since the Maxwell equation has not been used in the derivation of (6.3)—only the Dirac equation. Rather (6.3) is to be understood as a particular case of (3.3) in which, for $q > 0$, a radially inward pointing \bar{B} field arises due to the radially inward pointing spins of the Dirac field. (For either sign of q the \bar{B} field is aligned with the magnetic moments.) Thus, the distribution of dipole moments which is implicit in the Dirac field gives rise directly to a magnetic flux density which happens to be of monopole form, (6.3), though there is no monopole at the origin.

Comparison of (6.3) with (3.3) suggests that the amplitude of ψ may diverge $\propto 1/r$ sufficiently near the origin. This will indeed emerge from the numerical solutions below. Note, however, that although (6.3) must be the magnetic flux density everywhere, Eq. (3.3) is only part of the total \bar{B} . Consequently, the amplitude of ψ need not be $\propto 1/r$ everywhere. Indeed this would not be normalisable. Fortunately, we shall find that the amplitude of ψ falls off exponentially at large r .

Had we used the outward pointing spinor, (5.3), then (6.2) and (6.3) would change sign, i.e., the \bar{B} field is again aligned with the magnetic moments associated with the spins.

VII. THE RADIAL EQUATIONS FOR $F(r)$, $\lambda(r)$, AND $V(r)$

Dirac equation (2.1) can be expanded using (5.6) and the fact that $A^r = A^\theta = 0$ to give

$$\left\{ (E - qV) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 0 & \bar{\sigma} \cdot \bar{\nabla} \\ -\bar{\sigma} \cdot \bar{\nabla} & 0 \end{pmatrix} + q \begin{pmatrix} 0 & \sigma_\phi \\ -\sigma_\phi & 0 \end{pmatrix} A^\phi \right\} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{7.1}$$

This gives the following pair of simultaneous 2-spinor equations:

$$(E - qV - m)u + i\bar{\sigma} \cdot \bar{\nabla}v + q\sigma_\phi A^\phi v = 0, \tag{7.2}$$

$$(E - qV + m)v + i\bar{\sigma} \cdot \bar{\nabla}u + q\sigma_\phi A^\phi u = 0. \tag{7.3}$$

Using explicit expression (5.2) for the inward pointing spinor and (6.2) for A^ϕ the angular dependence cancels from (7.2) and (7.3) leaving a pair of equations in the radial coordinate only

$$\partial_r(\lambda F) = \left[(-E + qV + m) - \frac{\lambda}{r} \right] F, \tag{7.4}$$

$$\partial_r F = \left[(E - qV + m)\lambda - \frac{1}{r} \right] F. \tag{7.5}$$

In passing we note that substitution of (7.4) and (7.5) into (5.7) results in an identity, i.e., (5.7) is not an independent equation. However, since there are three unknown functions, F , λ , and V we need one more equation to form a soluble system. This is, of course, the Maxwell equation which so far has not been used. It is, in spherical polars,

$$\frac{1}{r^2} \partial_r (r^2 \partial_r) V = -4\pi q (1 + \lambda^2) F^2. \tag{7.6}$$

From this point on we adopt normalised variables, making the following replacements:

$$E/m \rightarrow E \quad mr \rightarrow r \quad \frac{q}{m} V \rightarrow V \quad \frac{q^2}{m^3} F^2 \rightarrow F^2. \tag{7.7}$$

Hence (7.4)-(7.6) become

$$\partial_r F = \left[(E + 1 - V) \lambda - \frac{1}{r} \right] F, \tag{7.8}$$

$$\partial_r \lambda = (1 + \lambda^2) (V - E) + (1 - \lambda^2), \tag{7.9}$$

$$\frac{1}{r^2} \partial_r (r^2 \partial_r) V = -(1 + \lambda^2) F^2. \tag{7.10}$$

VIII. ALTERNATIVE PARAMETERISATION

Radford, Ref. 16, worked in the van der Waerden representation of the Dirac bispinor and the gamma matrices. Radford’s bispinor can be written as

$$\begin{pmatrix} u_R \\ v_R \end{pmatrix} = \sqrt{R} \begin{pmatrix} e^{i\frac{\chi}{2}} \Phi \\ e^{-i\frac{\chi}{2}} \Phi \end{pmatrix}, \quad \text{where } \Phi = \begin{pmatrix} -e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} \end{pmatrix}, \tag{8.1}$$

where R and χ are functions of r only. Our bispinor, for the inward pointing spin case, can be obtained as

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = F \begin{pmatrix} \Phi \\ i\lambda\Phi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v_R + u_R \\ v_R - u_R \end{pmatrix} = \sqrt{2R} \begin{pmatrix} \Phi \cos \frac{\chi}{2} \\ -i\Phi \sin \frac{\chi}{2} \end{pmatrix}. \tag{8.2}$$

Hence,

$$F = \sqrt{2R} \cdot \cos \frac{\chi}{2} \quad \text{and} \quad \lambda = -\tan \frac{\chi}{2}. \tag{8.3}$$

In carrying out the numerical integrations, below, it is better to use the parameter χ rather than λ . This is because, in some cases, χ can pass through odd integer multiples of π , so that λ becomes singular (though λF , and hence the wavefunction, remain regular). For later convenience we also define

$$S = 2r^2 R = r^2 (1 + \lambda^2) F^2 \tag{8.4}$$

in terms of which the normalisation integral is simply

$$\int \psi^\dagger \psi \cdot d\tau = 4\pi \int_0^\infty S \cdot dr = 1. \tag{8.5}$$

In terms of the alternative parameter set S, χ, V the three radial equations (7.8)-(7.10) become

$$\partial_r S = -2S \sin \chi, \tag{8.6}$$

$$\partial_r \chi = 2(E - V - \cos \chi), \tag{8.7}$$

$$\partial_r (r^2 \partial_r) V = -S, \tag{8.8}$$

(8.6) and (8.8) are identical to the corresponding equations in Ref. 16, Eq. (2.8), when account is taken of the differing notation, namely, $\rho = 2r$, $a = -V$, $q = S$. However, in Radford’s notation (8.7) is $\partial_\rho \chi = E + a - \cos \chi$ whereas Ref. 16, Eq. (2.8), gives the equation in the form $\partial_\rho \chi = a - \cos \chi$. (Curiously the equation $\partial_\rho \chi = 1 + a - \cos \chi$ appears elsewhere in Ref. 16.) However, the equation

$\partial_\rho \chi = a - \cos \chi$ is equivalent to (8.7) provided that the appropriate boundary condition for a at infinity is adopted. This appears to be the intention in Ref. 16 since, in discussing the numerical results, the potential is stated to be $1 + a$ rather than a . However, since $V(r \rightarrow \infty) = 0$, (8.7) requires that under this interpretation the value of Radford's a at infinity should be E . But the asymptotic power series in $1/\rho$ given in Ref. 16 has $a(\rho \rightarrow \infty) = -1$, so that the solution of Ref. 16 would seem to be for the negative energy eigenvalue $E/m = -1$. This appears to be confirmed in Ref. 16 by the remark that *in the gauge for which $a \rightarrow 0$ as $\rho \rightarrow \infty$ the Dirac field has time dependence e^{imt} , i.e., $E = -m$.*

However, we are interested here in solutions with $E > 0$ and a very different picture from that of Ref. 16 emerges using Eqs. (8.6)-(8.8), or equivalently Eqs. (7.8)-(7.10), in this case. In particular, unlike Radford's solution, Ref. 16, these equations have no solution which is analytic in $1/r$ as $r \rightarrow \infty$ for positive E . This is proved next.

IX. THE ASYMPTOTIC BEHAVIOUR FOR $r \rightarrow \infty$ IS NOT ANALYTIC IN $1/r$

Normalisability requires $F^2 < O(r^{-3})$ as $r \rightarrow \infty$. If F is analytic in $1/r$ as $r \rightarrow \infty$, then we can put $LIM_{r \rightarrow \infty} F \rightarrow \tilde{F}r^{-\frac{n}{2}}$ for some constant \tilde{F} where $n > 3$. Integrating (7.10) then gives the asymptotic form of the potential as $r \rightarrow \infty$ to be

$$V \rightarrow \frac{Q}{r} - \frac{V_2}{r^{n-2}} + \text{higher order terms}, \tag{9.1}$$

where we have assumed the potential at infinity to be zero and put

$$V_2 = \frac{(1 + \lambda_\infty^2)\tilde{F}^2}{(n-3)(n-2)}, \tag{9.2}$$

where λ_∞ is the value of the λ parameter as $r \rightarrow \infty$. (9.1) allows for a point charge, q_0 , to exist at the origin, so that the net charge is $q_{net} = q + q_0$ and Q is related to the net charge by $Q = qq_{net}/4\pi$. If λ is also analytic in $1/r$ as $r \rightarrow \infty$ then we can write

$$\lambda = \lambda_\infty + \frac{\lambda_1}{r} + \frac{\lambda_2}{r^2} + O(r^{-3}). \tag{9.3}$$

Substituting (9.1) and (9.3) into (7.9) and equating coefficients of r^0 and r^{-1} gives

$$E = \frac{1 - \lambda_\infty^2}{1 + \lambda_\infty^2}, \tag{9.4}$$

$$Q = \frac{4\lambda_\infty\lambda_1}{(1 + \lambda_\infty^2)^2}. \tag{9.5}$$

Note that (9.4) is true providing only that as $r \rightarrow \infty$, $V \rightarrow 0$, and $\lambda \rightarrow \lambda_\infty$, a finite or zero constant. It does not depend upon the functional form of F, V, λ otherwise. This follows directly from (7.9). Using $LIM_{r \rightarrow \infty} F \rightarrow \tilde{F}r^{-\frac{n}{2}}$ in (7.8) and equating coefficients of the leading order term in $r^{-\frac{n}{2}}$ requires

$$\lambda_\infty(1 + E) = 0 \tag{9.6}$$

and hence, if $E \neq -1$,

$$\lambda_\infty = 0 \tag{9.7}$$

and hence that,

$$E = 1 \quad \text{and} \quad Q = 0. \tag{9.8}$$

However, even a solution with $E = 1$ and $Q = 0$ does not exist for our equation set if analytic behaviour in $1/r$ is assumed as $r \rightarrow \infty$, as we now show. Using (9.8) and equating coefficients of $r^{-\frac{n}{2}-1}$ in (7.8) requires

$$2\lambda_1 = 1 - \frac{n}{2}. \tag{9.9}$$

Including terms up to $1/r^2$ in (9.1) and (9.3) then equating coefficients of $1/r^2$ in (7.9) gives

$$\lambda_1 = \frac{2(\lambda_1^2 + 2\lambda_\infty\lambda_2)}{1 + \lambda_\infty^2} - 2\left(\frac{2\lambda_\infty\lambda_1}{1 + \lambda_\infty^2}\right)^2 + (1 + \lambda_\infty^2)V_2. \tag{9.10}$$

Putting $\lambda_\infty = 0$ this becomes

$$\lambda_1 = 2\lambda_1^2 + V_2 \quad \text{or} \quad V_2 = \lambda_1(1 - 2\lambda_1). \tag{9.11}$$

But (9.9) requires λ_1 to be negative since $n > 3$. It also requires $1 - 2\lambda_1 = \frac{n}{2}$ to be positive. Hence, it requires $\lambda_1(1 - 2\lambda_1)$ to be negative. But (9.11) then implies that V_2 is negative. But this is impossible because (9.2) shows V_2 to be positive definite since $n > 3$. We conclude that there is no local (normalisable) solution which is analytic in $s = 1/r$ as $r \rightarrow \infty$, for any energy, whether with a net charge or not, with the exception of the loophole at $E = -1$ exploited by Radford, Ref. 16. Our result is consistent with Ref. 6 which concluded that the solutions for ψ must be exponentially decaying at infinity.

X. ASYMPTOTIC EXPRESSIONS

In order to integrate Eqs. (7.8)-(7.10) or (8.6)-(8.8), suitable boundary conditions must be found. Here we state the asymptotic behaviour as $r \rightarrow \infty$ so that these expressions may be used as boundary conditions for subsequent numerical integration. The asymptotic form admits the possibility of an externally supplied point charge, q_0 , at the origin so that the net charge is $q_{net} = q + q_0$. In the numerical studies, positive, negative, and zero values of q_0 will be explored. Only the case $q_0 = 0$ is fully autonomous. From Eqs. (7.8)-(7.10) or (8.6)-(8.8) it is straightforward to derive the following asymptotic expressions valid as $r \rightarrow \infty$:

$$F \approx \tilde{F} e^{-z/2} (r^{\xi-1} + O(r^{\xi-2})), \tag{10.1}$$

$$\lambda \approx \lambda_\infty - (1 + \lambda_\infty^2) \frac{Q}{z} + O(z^{-2}) + (1 + \lambda_\infty^2)^2 \frac{\tilde{F}^2}{|\beta|^{1+2\xi}} \cdot e^{-z} \left(\frac{1}{2z^u} + O(z^{-u-1}) \right), \tag{10.2}$$

$$V \approx Q \left(\frac{1}{r} + O(e^{-z}) \right) - (1 + \lambda^2) \frac{\tilde{F}^2}{|\beta|^{2\xi}} \cdot \left\{ (1 - 2\xi) \left(1 - \frac{2\xi}{z} \right) E(u, z) + \frac{2\xi e^{-z}}{z^u} \right\}, \tag{10.3}$$

where

$$\xi = \frac{2EQ}{\beta} \quad \beta = \frac{4\lambda_\infty}{1 + \lambda_\infty^2}, \quad E = \frac{1 - \lambda_\infty^2}{1 + \lambda_\infty^2}, \quad Q = qq_{net}/4\pi, \tag{10.4}$$

$$z = -\beta r, \quad u = 2 - 2\xi, \quad E(p, z) = \int_z^\infty \frac{e^{-t}}{t^p} dt, \tag{10.5}$$

and \tilde{F} is a constant to be found by normalisation of the wavefunction. The exponential integral defined by (10.5) has the following asymptotic expansion:

$$E(p, z) \approx \frac{e^{-z}}{z^p} \left(1 - \frac{p}{z} + \frac{p(p+1)}{z^2} - \frac{p(p+1)(p+2)}{z^3} + \dots \right) \tag{10.6}$$

(approximately valid only for $z \gg p$).

The terms in Q in (10.2) and (10.3) are of leading order z^{-1} and hence dominate the terms in \tilde{F}^2 which are of leading order $z^{-u} e^{-z}$. For this reason we have neglected cross terms in $Q\tilde{F}^2$ since they are negligible compared to the Q terms when $Q \neq 0$ and obviously can be dropped when $Q = 0$. The \tilde{F}^2 terms are included to cater for the case $Q = 0$ and also because the Coulomb term, Q/r , gives zero on the LHS of (7.10) or (8.8).

A particular solution is specified by choosing values for E and Q . The asymptotic amplitude factor \tilde{F} is determined by normalising the solution to unity, Eq. (8.5). The most notable feature of the asymptotic behaviour is that the wavefunction is exponentially decaying at infinity, as long as $\beta < 0$, consistent with Ref. 6. The potential is also exponentially decaying at infinity if $q_{net} = 0$ else

it decays as the Coulomb field of the net charge, as expected. The parameter λ takes the finite value λ_∞ at infinity. To be normalisable as $r \rightarrow \infty$ it is sufficient that $\beta < 0$ and hence $\lambda_\infty < 0$. The parameter ξ may be of either sign depending upon the sign of Q . The case $\lambda_\infty = \beta = 0$, $E = 1$, $Q \neq 0$ is pathological in (10.1), since $|\xi| \rightarrow \infty$. The case $\lambda_\infty = \beta = 0$, $E = 1$, $Q = 0$ fails to be normalisable. Hence, positive energy requires that $-1 < \lambda_\infty < 0$, hence $0 < \chi_\infty < \pi/2$ and $0 < E < 1$. [In passing we note that the solution of Ref. 16 has $\chi_\infty = \pi$, hence $\lambda_\infty \rightarrow -\infty$ consistent with $E = -1$.]

Note that at this stage it is not clear that the solution which behaves at large r as given by (10.1)-(10.6) is normalisable because the behaviour at small r has not yet been found. This will follow from the numerical integration.

XI. NUMERICAL INTEGRATION

The numerical integration is carried out by splitting the Maxwell equation, (7.10) or (8.8), into a pair of first order equations. The equation sets (7.8)-(7.10) or (8.6)-(8.8) then become four first order equations. A suitably large r_{\max} is chosen at which to apply boundary conditions (10.1)-(10.6). Numerical integration using a 4th order Runge-Kutta algorithm was carried out by shooting back towards small r . Because the solution is divergent at the origin some minimum radius must be specified at which to terminate the integration, r_{\min} . This must be small enough to provide an accurate normalisation. In most of the results presented $r_{\min} = 10^{-4}$ and $r_{\max} = 10$, using 10 000 unequal integration steps, consecutive steps being reduced in size by a constant factor of typically 0.9995. The asymptotic amplitude factor \vec{F} is initially given an arbitrary value. The integration is repeated with an improved estimate of \vec{F} until the normalisation integral, (8.5), is unity to within some tolerance (typically 10^{-4}). The non-linearity of the equations prevents \vec{F} being found by simple scaling.

Checks on the numerical stability of the integration procedure were carried out as follows. (i) Repeat integrations using different values for r_{\min} or r_{\max} produced the same results to within acceptable tolerance. (ii) Repeat integrations using larger numbers of integration steps produced the same results to within acceptable tolerance. (iii) Integration was carried out using equation set (7.8)-(7.10) as well as equation set (8.6)-(8.8), obtaining the same results. (iv) The values of F, λ, V at r_{\min} were used as the boundary conditions for a second integration, shooting back to r_{\max} . The original boundary conditions at r_{\max} based on (10.1)-(10.6) were regained within acceptable tolerance. (v) The derivatives $\partial_r F$ and $\partial_r \lambda$ were estimated from the numerical solution for F and λ by finite difference and used within Eqs. (7.8) and (7.9) to calculate the energy, E , confirming it agreed at all positions r with the initial assumption.

Integrations assumed $q = 1$, i.e., the Dirac field of a positron, and inward pointing spins, consistent with (5.2) and (8.2), and hence with an inward pointing magnetic flux density, (6.2). Runs were

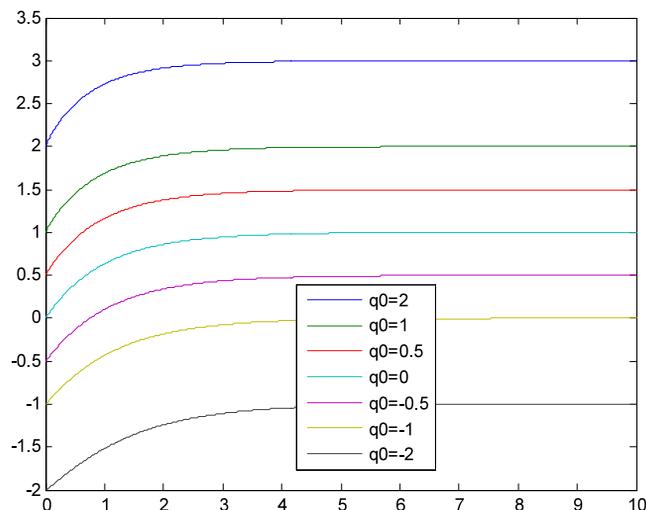


FIG. 1. The charge within a sphere of radius r versus r ($E = 0.9$).

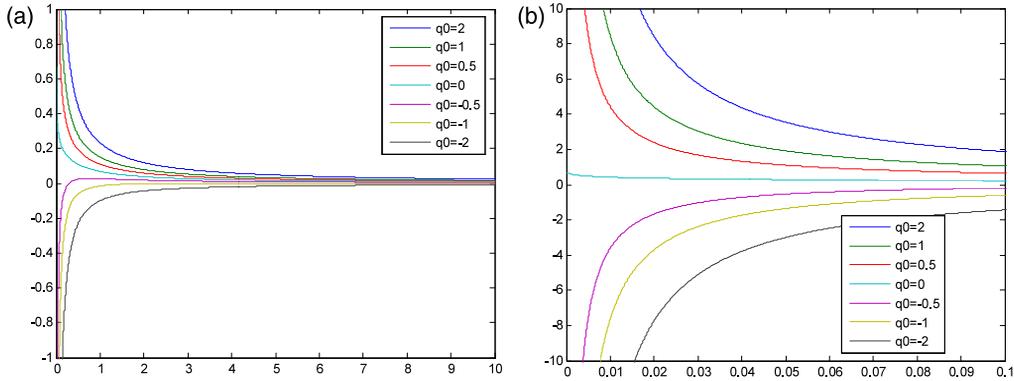


FIG. 2. (a) The potential, V , against r (large r), $E = 0.9$. The uppermost curve has $q_0 = 2$, progressing to the lowest curve with $q_0 = -2$. (b) The potential, V , against r (small r), $E = 0.9$. The uppermost curve has $q_0 = 2$, progressing to the lowest curve with $q_0 = -2$.

carried out initially for $E = 0.9$ and for central charges, q_0 , of 2, 1, 0.5, 0, -0.5, -1, and -2. The solutions were well behaved and normalisable in all cases. Figures 1–5 are graphs of the following quantities: Fig. 1: The charge within a sphere of radius r , plotted against r ; Figs. 2(a) and 2(b): The potential plotted against r . For the case $q_0 = 0$, the potential is only logarithmically divergent at the origin, as will be shown shortly. For $q_0 \neq 0$, the potential tends to Coulomb form as $r \rightarrow 0$ and hence diverges as $1/r$; Figs. 3(a) and 3(b): The parameter χ plotted against r . Its value at infinity, 0.451, is determined by the assumed energy, $E = 0.9$, via (8.3) and (10.4). Note that for $q_0 = 2$ χ exceeds π at small r so λ passes through a singularity. This does not affect the integration if carried out using χ rather than λ ; Fig. 4(a): The wavefunction squared-amplitude parameter R , defined in (8.1)–(8.4), plotted against r , both on logarithmic scales. That the curves become straight lines below roughly $r \sim 1$ implies that the wavefunction amplitude is a power of r in that region. Inspection of Fig. 4(a) shows that this dependence is roughly $R \propto r^{-2}$; Fig. 4(b): The wavefunction squared-amplitude parameter R plotted against r , with R on a logarithmic scale but r linear. That the curves become straight lines above roughly $r \sim 1$ implies that the wavefunction amplitude becomes dominated by an exponential dependence upon r for large r , consistent with the asymptotic form of (10.1); Fig. 5: The parameter $S = 2r^2R$ plotted against r . This parameter has a finite value at the origin, showing that $R \propto r^{-2}$ becomes exact for sufficiently small r . This is the result which we anticipated in Sec. VI, consistent with the interpretation of the monopole-like magnetic flux density arising from the spin of the Dirac field via Eq. (3.3).

A series of runs was also carried out for zero central charge, $q_0 = 0$, and energies of 0.01, 0.1, 0.5, 0.9, 0.99, and 0.999. For the largest and smallest energies r_{\max} was increased, to a maximum

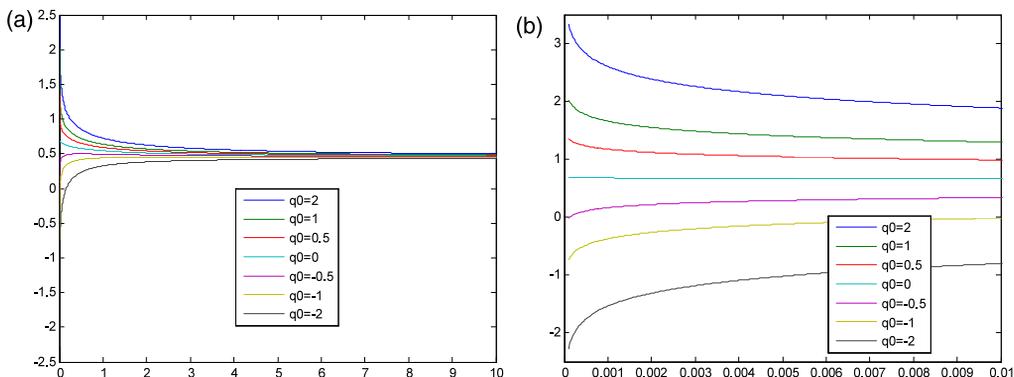


FIG. 3. (a) The phase parameter χ plotted against r (large r), $E = 0.9$. The uppermost curve has $q_0 = 2$, progressing to the lowest curve with $q_0 = -2$. (b) The phase parameter χ plotted against r (small r), $E = 0.9$.

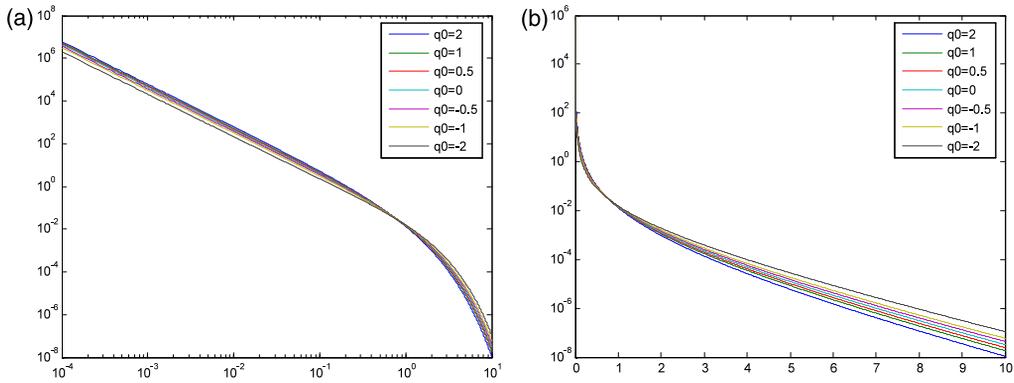


FIG. 4. (a) The squared-amplitude R against r (both scales logarithmic), $E = 0.9$. For small r the uppermost curve has $q_0 = 2$, and the lowest curve has $q_0 = -2$; for large r this order reverses. (b) The squared-amplitude R against r (R logarithmic, r linear), $E = 0.9$.

of 500, to ensure that the value of λ at r_{\max} was close to λ_∞ . The solutions were well behaved and normalisable for the whole energy range $0.01 \leq E \leq 0.999$. Results are presented as follows: Fig. 6: The charge within a sphere of radius r , plotted against r ; Fig. 7(a): The potential plotted against r ; Fig. 7(b): The potential plotted against r with r on a logarithmic scale, showing clearly the logarithmic singularity at the origin for $q_0 = 0$; Fig. 8: The parameter χ plotted against r . Its value at infinity is determined by the assumed energy, $E = 0.01$ corresponds to $\chi_\infty = 1.5608$ and $E = 0.999$ corresponds to $\chi_\infty = 0.04473$; Fig. 9(a): The wavefunction squared-amplitude parameter R , defined in (8.1)-(8.4), plotted against r , both on logarithmic scales. Again the curves become straight lines below roughly $r \sim 1$ implying that $R \propto r^{-2}$ for small r ; Fig. 9(b): The wavefunction squared-amplitude parameter R plotted against r , with R on a logarithmic scale but r linear. The curves become straight lines above roughly $r \sim 1$ implying that the wavefunction amplitude becomes dominated by an exponential dependence upon r for large r , consistent with (10.1); Fig. 10: The parameter $S = 2r^2R$ plotted against r . This parameter has a finite value at the origin for all energies, showing that $R \propto r^{-2}$ becomes exact for sufficiently small r .

It is worth emphasising that the finite value of S at the origin, S_0 , is related to the logarithmic singularity in the potential for $q_0 = 0$, direct integration of (8.8) for sufficiently small r giving $V \approx -S_0 \log r$.

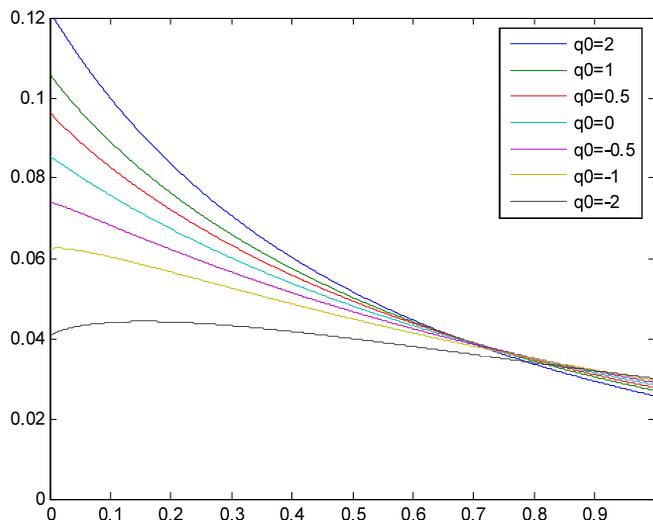


FIG. 5. $S = 2r^2R$ for small r showing that S is finite at $r = 0$ ($E = 0.9$). For small r the uppermost curve has $q_0 = 2$, and the lowest curve has $q_0 = -2$; for large r this order reverses.

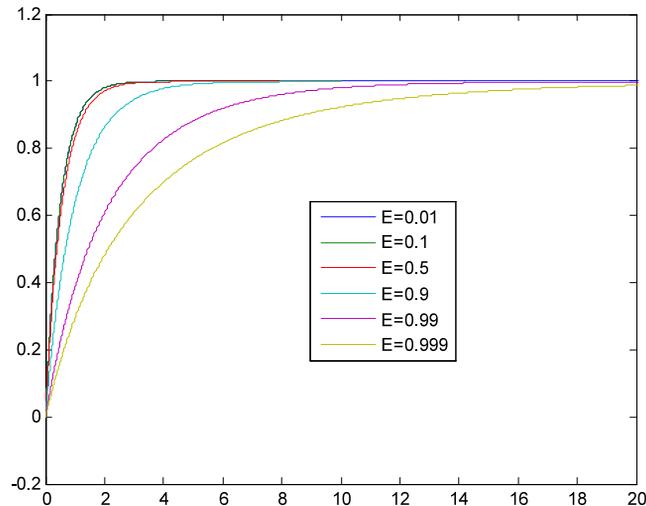


FIG. 6. The charge within a sphere of radius r versus r ($q_0=0$). The uppermost curve has $E = 0.01$, progressing to the lowest curve with $E = 0.999$.

XII. WHY ARE THE SOLUTIONS BOUND?

Bound solutions with an externally imposed central charge with opposite sign to the field charge, $qq_0 < 0$, can readily be understood as resulting from the Coulomb attraction. However, it is initially surprising that there is a bound solution with $q_0 = 0$ for which the Dirac field electrostatic self-repulsion might be expected to prevent a bound state. Still more surprising is the result that there are bound solutions with $qq_0 > 0$ since the central charge now increases further the Coulomb repulsion. Unlike Lisi, Ref. 14, and Comech and Stuart, Ref. 10, who were dealing with negative energy solutions, we do not need to resort to anything so exotic as the Klein paradox to explain the bound nature of our positive energy solutions. A more prosaic explanation is ready at hand in the form of the attractive magnetic forces. We have seen that, for the case of inward pointing spins, and taking $q > 0$, the net effect of the Dirac field is to produce a radially inward pointing magnetic flux density, \vec{B} , (6.2). The classical energy of a permanent magnetic dipole, \vec{m} , in a magnetic flux density, \vec{B} , is $-\vec{m} \cdot \vec{B}$. Consequently, since \vec{m} and \vec{B} are both pointing inwards in this case, the magnetic energy is negative. If this negative magnetic energy is larger in magnitude than the positive electrostatic energy which occurs for $q_0 \geq 0$, then the total potential energy is negative and a bound state becomes a possibility.

A rough classical model suffices to demonstrate that the negative magnetic energy can always dominate the positive electrostatic energy, even for $qq_0 > 0$. Suppose the Dirac field is replaced with

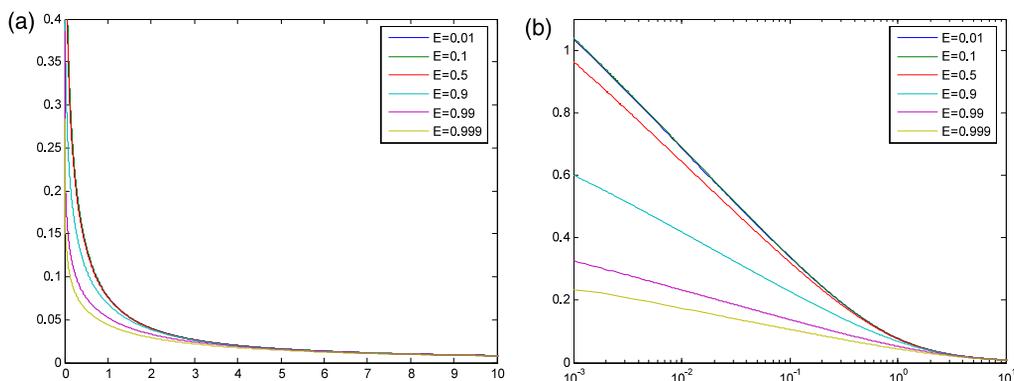


FIG. 7. (a) The potential, V , against r (large r), $q_0=0$. The uppermost curve has $E = 0.01$ – 0.5 , progressing to the lowest curve with $E = 0.999$. (b) The potential, V , against r at small r for $q_0 = 0$ showing its logarithmic singularity. The uppermost curve has $E = 0.01$ – 0.1 , progressing to the lowest curve with $E = 0.999$.

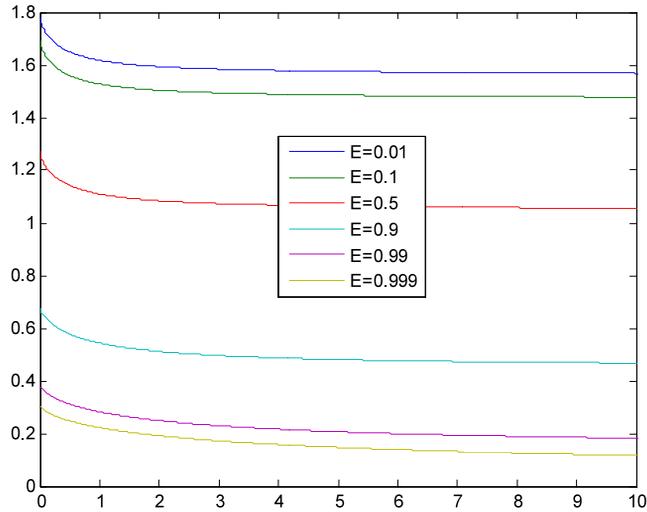


FIG. 8. The phase parameter χ plotted against r (large r), $q_0=0$. The uppermost curve has $E = 0.01$, progressing to the lowest curve with $E = 0.999$.

a classical distribution of charge and a distribution of radially inward dipole moments whose densities are both proportional to $1/r^2$ for $r < a$ and zero for $r > a$. The latter is a crude proxy for the exponential decay at larger r (Figs. 4 and 9). The classical electrostatic and magnetostatic energies are divergent at the origin, but truncating the integral of energy density at some $r = \varepsilon \ll a$ the total electromagnetic energy can be evaluated to be

$$E_{classical}^{e/m} = \frac{q^2}{2\pi a} + \frac{qq_0}{4\pi a} \log \frac{a}{\varepsilon} - \frac{1}{2am} \left(\frac{1}{\varepsilon} - \frac{1}{a} \right). \tag{12.1}$$

The first term is the electrostatic self-energy of the distributed charge (q), the second term is the electrostatic energy of the distributed charge in the field of the central point charge, q_0 , and the third term is the magnetostatic self-energy of the distributed dipole moments for an integrated *magnitude* of dipole moment of q/m and a magnetic flux density given by (6.3). Providing ε is taken as sufficiently small compared with both a and $1/m$, (12.1) will be negative however large qq_0 may be. (12.1) ignores the self-energy of the central point charge alone. This may be considered as renormalised, or, alternatively, it may be formally included and the derivative of (12.1) taken with respect to a whereupon the infinite self-energy of the point charge disappears. That this derivative is positive for sufficiently small ε demonstrates that the classical force is inwards (attractive).

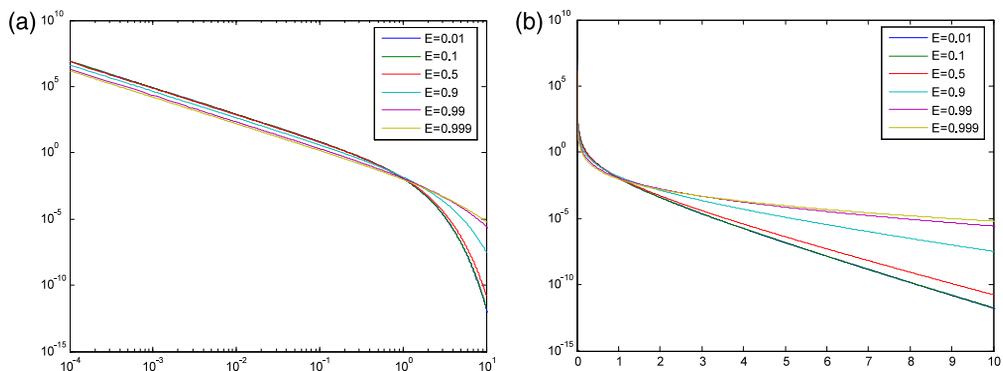


FIG. 9. (a) The squared-amplitude R against r (both scales logarithmic), $q_0=0$. For small r the uppermost curve has $E = 0.01$, and the lowest curve has $E = 0.999$; for large r this order reverses. (b) The squared-amplitude R against r (R logarithmic, r linear), $q_0=0$. For $r > 1$ the uppermost curve has $E = 0.999$, and the lowest curve has $E = 0.01$.

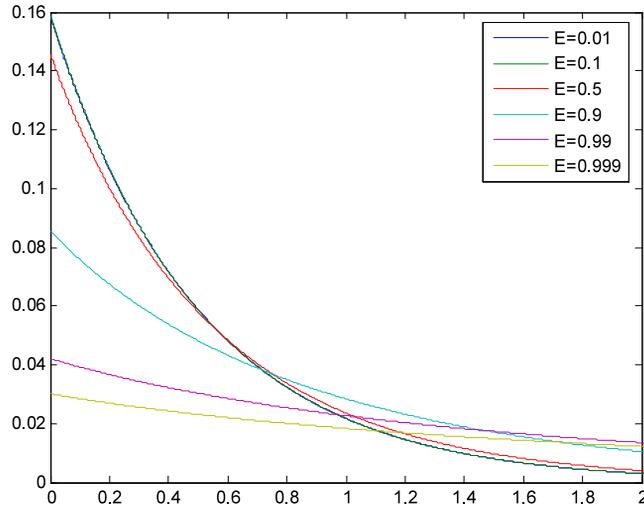


FIG. 10. $S = 2r^2R$ for small r showing that S is finite at $r = 0$ ($q_0 = 0$). For small r the uppermost curve has $E = 0.01$ – 0.1 , and the lowest curve has $E = 0.999$; for larger r this order reverses.

This classical model is pathological due to the singularity as $\varepsilon \rightarrow 0$. The Dirac version avoids this pathology. Eq. (7.4) or (7.9) can be re-written in normalised form as

$$E = 1 + V - \left\{ \frac{\lambda}{F} \partial_r F + \partial_r \lambda + \frac{\lambda}{r} \right\} = 1 + V - \left\{ \frac{2\lambda^2 + \partial_r \lambda}{1 + \lambda^2} \right\}. \tag{12.2}$$

The first term on the RHS of (12.2), i.e., unity, is the rest mass. The second term, V , is the electrostatic energy. So we may associate the term in $\{ \}$ with the magnetic energy. (12.2) is an identity at large r if (10.1)–(10.3) are substituted. For $r \rightarrow 0$ the singularities in the individual terms in (12.2) cancel. As a consistency check, the numerical solutions were used to evaluate (12.2), confirming that the assumed energy was reproduced at all points, r .

Expression (12.2) for the energy eigenvalue of the Dirac field can also be derived directly from the energy-momentum tensor, i.e., from

$$E = \frac{\bar{\psi} (-\vec{\gamma} \cdot [i\vec{\nabla} + q\vec{A}] + m + V\gamma^0) \psi}{\psi^+ \psi}. \tag{12.3}$$

Substitution of (5.2) and (5.6) into (12.3) reproduces (12.2). It should be noted that throughout the energy E has been identified with the eigenvalue of the operator $i\partial_t$. We have not discussed the energy within the electromagnetic field *per se*. In the case that a point charge is externally imposed at the origin, the electrostatic field energy will, of course be divergent. Of greater interest is the case with $q_0 = 0$. In this case V is only logarithmically divergent at the origin, corresponding to a finite integrated electrostatic field energy, $2\pi \int |\partial_r V|^2 r^2 dr$. By virtue of $\nabla^2 V = -\rho$, where ρ is the charge density, this expression for the electrostatic field energy is the same as $2\pi \int V \rho r^2 dr$, and this form for the electrostatic energy is reproduced by the V term in (12.2) or (12.3) when the matrix element is calculated.

XIII. CONCLUSIONS

Stationary, static, spherically symmetric solutions of the combined Maxwell and Dirac equations have been found which are localised and normalisable to unity. The solutions apply to any bound energy in the range $0 < E < 1$, where unity represents the bare mass in the Dirac equation. A point charge of any magnitude and either sign may be placed at the origin and the solutions remain well behaved. However, no such central charge is necessary to result in a bound solution. Note that this contradicts the claim in Ref. 17 that an isolated, stationary, static Maxwell-Dirac system must be electrically neutral.

The magnetic flux density is equal to that of a monopole at the origin. However, no monopole is present, the magnetic flux being a result of the dipole moment distribution of the Dirac field. Hence, if $q > 0$ and the Dirac field spins are chosen to be radially inwards, the \vec{B} field which results is also radially inwards. The resulting magnetic self-energy of the Dirac field is therefore negative and it is this which results in a bound state ($E < 1$). The case which omits any central point charge is therefore a self-sustaining bound state solution of the Maxwell-Dirac system which is localised and has finite energy and which requires no externally added features (i.e., it is a soliton). As far as the author is aware, this is the first time that such an exact, self-consistent solution with positive energy eigenvalue has been reported. However, the solution is not unique since the energy is arbitrary within the range $0 < E < 1$.

Why are such solutions not physically realised? There are (at least) four possible reasons. The first is the possibility that, in fact, the Dirac equation has no classical meaning. The second possibility is that the assumed spherical symmetry is unstable against collapse to an asymmetric state. The third possibility is that the solutions with $E > 0$ would be unstable against radiative transitions to a state tending towards $E \rightarrow 0$ by emission of photon pairs. The final possibility is an appeal to a self-consistency principle beyond the mathematics of the equations *per se*, namely, that only those solitons may exist which reproduce the properties of the Dirac field quantum. This would require an axisymmetric solution which correctly reproduces the angular momentum and dipole moment of the Dirac field, which is clearly impossible in a spherically symmetric solution for which these quantities are zero.

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