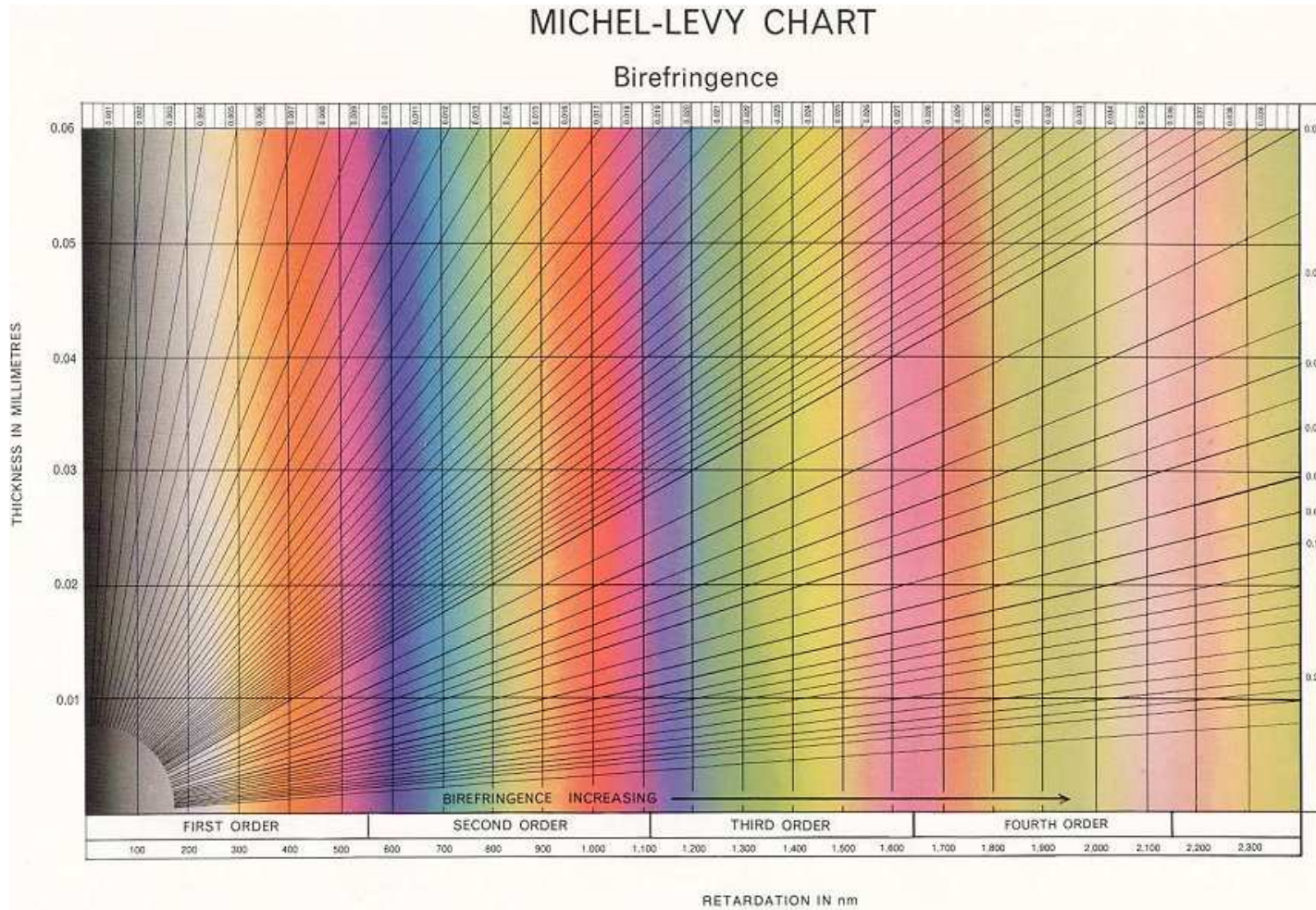


This brief note explains why the Michel-Levy colour chart for birefringence looks like this...

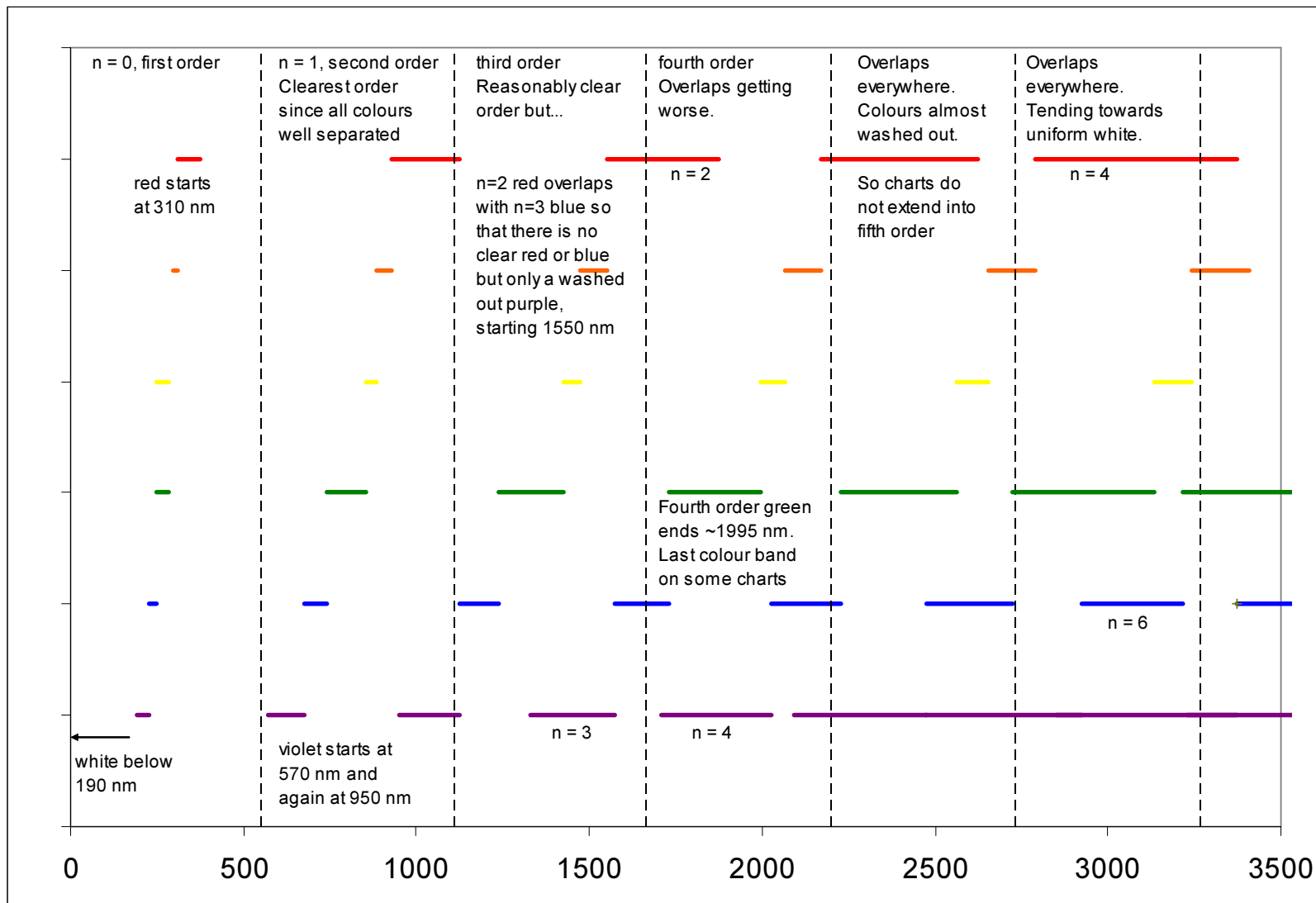


Theory of Levy Colour Chart for Birefringent Materials Between Crossed Polars

- **Birefringence** = $n_2 - n_1$, the difference of the refractive indices for the two rays
- Frequency of rays is unchanged in material, $f = \omega / 2\pi$, ω = angular frequency
- Wavenumber of rays changes to n_1k and n_2k where $k = \frac{2\pi}{\lambda}$ is the wavenumber in vacuum and λ is the wavelength in vacuum.
- So the wavelengths are shorter in the material, λ / n_1 and λ / n_2
- Wave amplitude in vacuum proportional to $\exp i\{kx - \omega t\}$.
- Ray amplitudes in material proportional to $\exp i\{n_1kx - \omega t\}$ and $\exp i\{n_2kx - \omega t\}$
- The phase of the wave **in vacuum** would be greater by an amount kx at a distance of x (at the same time)
- So a phase difference can be converted to an equivalent distance **in vacuum** by dividing by k
- The two rays at the same time and at the same position in the material differ in phase by $(n_2 - n_1)kx$.
- This is equivalent to a distance **in vacuum** of $(n_2 - n_1)x$.
- Hence the **retardation** $r = (n_2 - n_1)x$ is the **distance in vacuum** which would cause the same phase difference as the two rays experience after passing a real distance of x through the material.
- Lines on the Levy chart are x (thickness) up the y-axis versus r , which therefore has a slope of $1/(n_2 - n_1)$
- Rationalisation of the Fringe Colours (see Figure, next page):-
- Peaks in colours occur when $r = (n + 0.5)\lambda$, for $n = 0, 1, 2, \dots$ (see last page for why) so these colours are expected to be found in the following r -ranges,

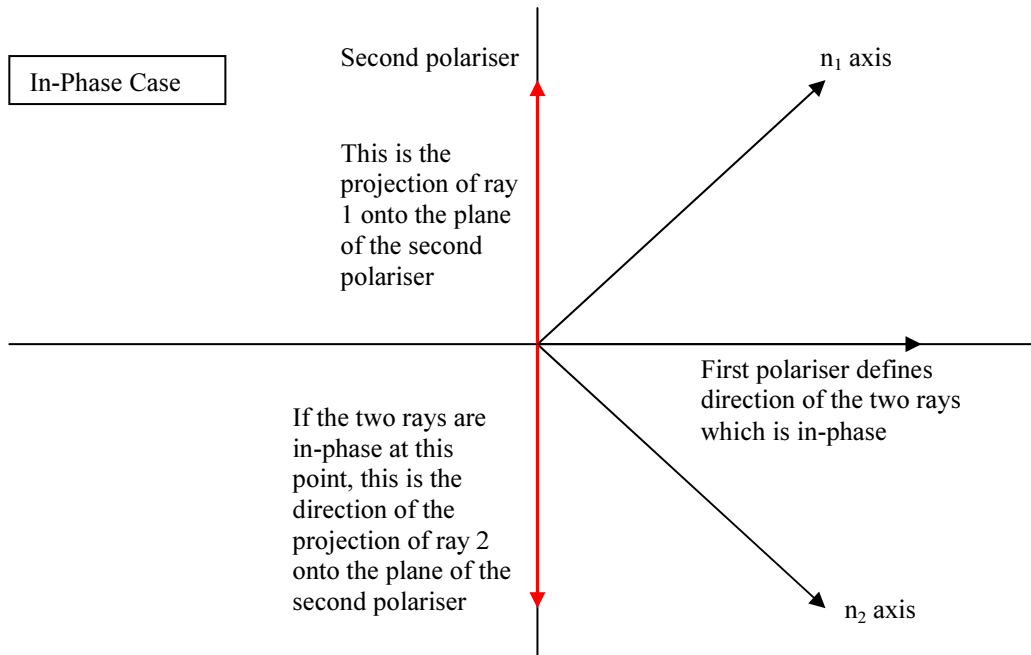
	λ from, nm	λ to, nm	r $n = 0$ nm		r $n = 1$ nm		r $n = 2$ nm		r $n = 3$ nm	
violet	1	380	190	225	570	675	950	1125	1330	1575
blue	2	450	225	247.5	675	742.5	1125	1237.5	1575	1732.5
green	3	495	247.5	285	742.5	855	1237.5	1425	1732.5	1995
yellow	4	570	285	295	855	885	1425	1475	1995	2065
orange	5	590	295	310	885	930	1475	1550	2065	2170
red	6	620	310	375	930	1125	1550	1875	2170	2625

- This gives a reasonable representation of the chart (see diagram on next page)
- It is white below 190 nm
- It blurs out due to overlaps at fourth order and beyond
- The reason why there are no narrow violet/blue/green bands in the first order range 190-285 nm is explained on the last page



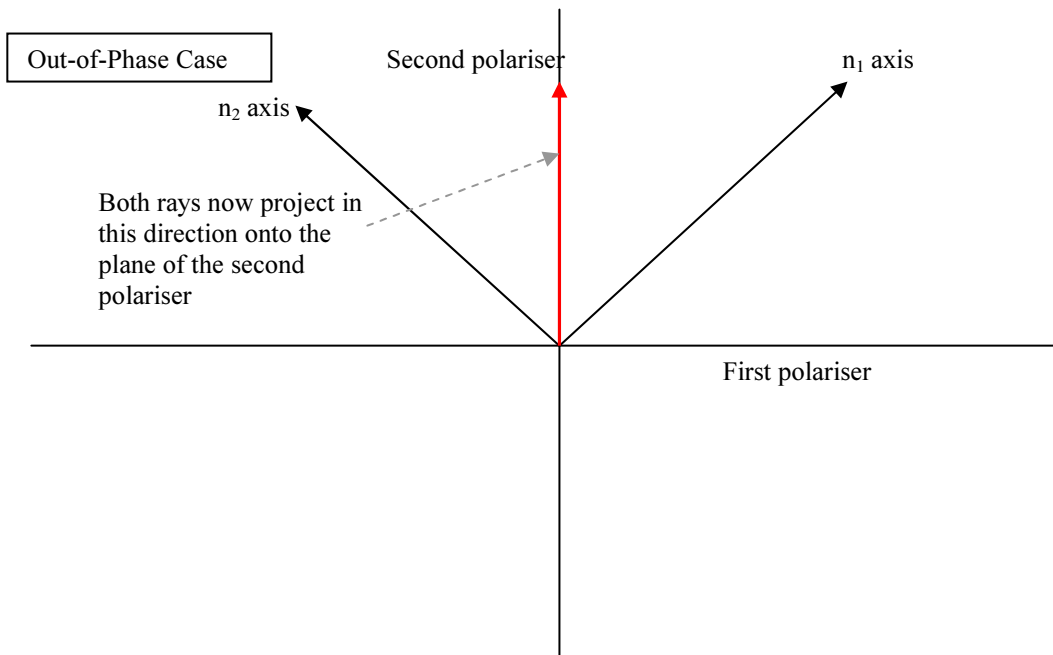
Why are the peaks at $r = (n + 0.5)\lambda$ not at $r = n\lambda$?

This is because of the crossed-polars.



Hence, if the rays are in-phase, i.e., if $r = n\lambda$ for $n = 0, 1, 2, \dots$ then the waves destructively interfere because their projections onto the plane of the second polariser are in opposite directions.

Conversely, if the rays are 180° out-of-phase, i.e., if $r = (n + 0.5)\lambda$ for $n = 0, 1, 2, \dots$ then the waves constructively interfere because ray 2 is now pointing in the opposite direction and the projections of the two rays onto the plane of the second polariser are now in the same direction, thus....



How does this work algebraically?

The ray amplitudes in the material are proportional to $\exp i\{n_1 kx - \omega t\}$ and $\exp i\{n_2 kx - \omega t\}$ respectively. The two waves combine with a relative minus sign so the total amplitude is,

$$\exp i\{n_2 kx - \omega t\} - \exp i\{n_1 kx - \omega t\}$$

But a wave intensity is proportional to the absolute square of its amplitude, which is,

$$\begin{aligned} & (\exp i\{n_2 kx - \omega t\} - \exp i\{n_1 kx - \omega t\})(\exp -i\{n_2 kx - \omega t\} - \exp -i\{n_1 kx - \omega t\}) \\ &= 2 - 2 \cos(n_2 - n_1)kx \\ &= 4 \sin^2 \frac{(n_2 - n_1)}{2} kx \end{aligned}$$

The peaks in $\sin^2 \theta$ occur at $\theta = (n + 0.5)\pi$ for $n = 0, 1, 2, \dots$

So the peaks in a given colour, of wavelength λ , occur at,

$$\frac{n_2 - n_1}{2} kx = \frac{n_2 - n_1}{2} \cdot \frac{2\pi}{\lambda} x = (n + 0.5)\pi \quad \text{i.e., for } r = (n_2 - n_1)x = (n + 0.5)\lambda$$

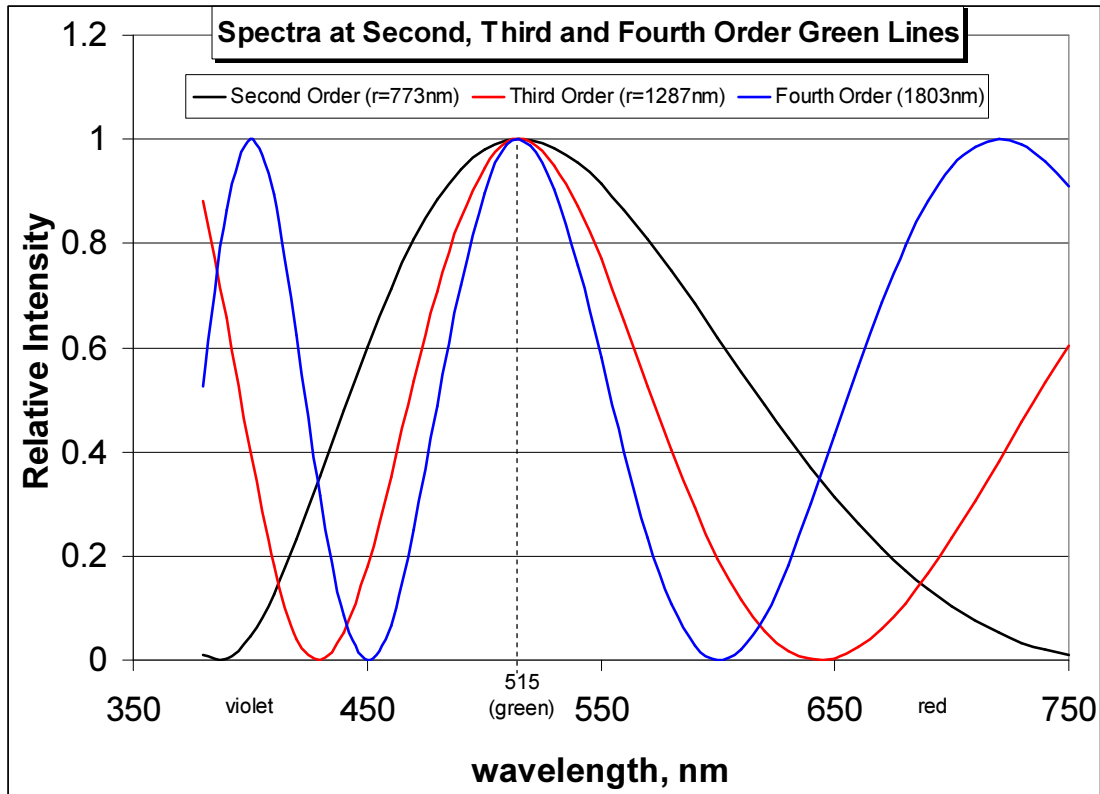
How could the colour chart be calculated more accurately?

Why does the character of the colours change order-by-order?

The spectrum (the intensity of light at wavelength λ) is given as derived above by

$\sin^2 \frac{(n_2 - n_1)}{2} kx = \sin^2 \frac{(n_2 - n_1)\pi x}{\lambda} = \sin^2 \frac{\pi r}{\lambda}$. This is plotted against λ below for the case of the green lines at orders 2, 3 and 4, namely at $r = 773, 1287$ and 1803 nm, all of which correspond to a peak at $\lambda = 515$ nm (green), as can be checked by dividing r by 1.5, 2.5 and 3.5 respectively. Accordingly the spectrum shown below reaches a peak at $\lambda = 515$ in all three cases. However the spectra are otherwise very different.

- The second order spectrum has a large contribution from around the green wavelength only – corresponding to quite a pure green line at second order;
- The third order spectrum has a substantial amount of violet and red as well as green, corresponding to a less pure green at third order;
- The fourth order spectrum has as much violet and red as green, corresponding to a particularly washed-out green at fourth order.



Exactly the same method for the first order where a green line might have been expected, i.e., for $r = 0.5 * 515 = 257 \text{ nm}$, produces the almost flat spectrum given below. This explains why, in the first order, where green (or violet or blue) lines might have been expected, at 190-285 nm, there are none - but rather just white.

