Where is the Charge in the Kerr-Newman Solution?

The charge is zero or finite but divergent in parts according to choice

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1. Summary

We are concerned here with the solution of the Einstein equations for a spinning, charged mass in the case that $m < \sqrt{a^2 + q^2}$. This is the Kerr-Newman solution with a naked singularity and would be appropriate for (say) and electron or a proton if quantum mechanics did not intervene. The topology of the Kerr-Newman spacetime in this regime is a ring which borders a disc, the latter acting as a bridge (or wormhole) between regions of positive and negative 'radius'. The ring is a real curvature singularity and also a singularity of the electromagnetic field. The curvature and the electromagnetic field are regular on the disc away from its edge.

Confining attention to the physical spacetime with positive 'radius', we present plots of the electric field vector which illustrate that the charge resides on both the ring and the disc, but with opposite signs. The charge seen at infinity (q) is the algebraic sum of these two opposing contributions. Integration shows that the magnitude of the charge on the disc is, in fact, divergent. Hence, the finite charge seen at infinity is effectively a result of an auto-renormalisation within the Kerr-Newman solution: the sum of the disc and ring contributions to the charge involves an infinite cancellation to leave a finite difference.

However, the more fundamental observation is that, with respect to the maximal analytical extension of the spacetime, i.e., including the region with negative 'radius', there actually is no charge at all. This follows by evaluating the 4-current density as the covariant derivative of the electromagnetic field tensor – which is identically zero. The same result can also be deduced by tracking the electric field lines. The charge which apparently resides on the disc disappears when the field is allowed to pass through the 'bridge' into the region with r < 0. Similarly, the charge on the ring disappears since the same number of lines of force emerge from it into the region with r > 0 as impinge upon it from the region with r < 0. All lines of force are therefore continuous at all finite points, and hence there is no charge. The lines of force which are seen at $r = +\infty$ are balanced by those originating at $r = -\infty$. Adopting a topology in which the r > 0 points at infinity are joined to those in the r < 0 region means that the field lines are continuous everywhere.

2. The Kerr-Newman Solution

In Boyer-Lindquist coordinates, the Kerr-Newman solution is,

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta \cdot d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2 + a^2\right)d\phi - a\,dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \quad (1)$$

where, $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2 + q^2$ (2)

In (1) the coordinates r, θ, ϕ, t can be interpreted as contravariant, $\{x^\mu\}$, and the coefficients are the covariant components of the metric tensor, $\{g_{\alpha\beta}\}$. The quantities m, a, q are the geometrical mass, spin and charge respectively. They are defined to have units of length. If the usual physical values of these quantities are written M,J,Q then we have,

$$q^2 = \frac{G}{\varepsilon_0 c^4} Q^2 , \qquad m = \frac{G}{c^2} M , \qquad a = \frac{GJ}{mc^3} = \frac{J}{Mc} \tag{3}$$

In the case of an electron, say, which has $J = \hbar / 2$, we find,

$$q = 4.88 \times 10^{-36}\,\text{m}, \qquad m = 6.75 \times 10^{-58}\,\text{m}, \qquad a = 1.93 \times 10^{-13}\,\text{m}$$

(in metres) and hence $m << q << a$ so that the electron has a naked singularity (in classical physics)[1].

The electromagnetic 4-vector potential consistent with the Kerr-Newman solution is,

$$\overline{A} = -\frac{qr}{\rho^2}\left(\overline{d}t - a \sin^2\theta \cdot \overline{d}\phi\right) \tag{4}$$

Care is required in interpreting the electric and magnetic field components. In Boyer-Lindquist coordinates the covariant components of the field $F_{rt} = E_r$ and $F_{\theta t} = E_\theta$ are generally written,

$$E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \qquad \text{and} \qquad E_\theta^{BL} = -2qa^2 r \sin\theta \cos\theta / \rho^4 \tag{5}$$

However, it is immediately apparent that there is something funny about E_θ^{BL} because the dimensions are not as expected (the electric field usually has dimensions charge/area). This is because it has been defined with respect to $\overline{d}\theta$ rather than the more usual unit vector with unit length. In spherical polars in Euclidean space the appropriate unit vector would be $r \cdot \overline{d}\theta$. Here we interpret the spatial part of the Boyer-Lindquist coordinates as an ellipsoidal coordinate system which are related to a 'Cartesian' system as follows,

$$z = r\cos\theta , \qquad \eta = \sqrt{a^2 + r^2}\sin\theta , \qquad x = \eta\cos\phi , \qquad y = \eta\sin\phi \tag{6}$$

Of course, x, y, z is not really Cartesian, since the underlying spacetime is curved.

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[1] Note that any 'point' particle with spin has a naked singularity if its mass is less than $\sqrt{\frac{\hbar c}{2G}}$ = 1.54 x 10^{-8} kg = 15 µg. This fate is avoided by normal matter since, at (say) the density of water, such a mass would be of macroscopic size, namely a cube of side 0.25mm. But, with a reasonable angular momentum, the geometric spin, a, will be much smaller than this. Since the source occupies a much greater spatial region than a, the 'point' source Kerr-Newman solution is not applicable. To make the geometric spin exceed 0.25mm for a mass of 15 µg requires an angular momentum exceeding 0.001 kgm^{2}s^{-1}. Since the moment of inertia of such a mass is ~2 x 10^{-16} kgm^2, a rotational rate of ~5 x 10^{12} cycles per second is needed, resulting in an edge velocity which would exceed the velocity of light. So a naked singularity is effectively vetoed for matter of familiar density. Similarly, any 'point' particle which has charge has a naked singularity if its mass exceeds $\sqrt{4\pi\alpha\frac{\hbar c}{G}}$ = 0.66 x 10^{-8} kg, i.e. of a similar magnitude. But the length scale which determines whether the situation qualifies as a 'point source' is still determined by the geometrical spin, a, and hence is vetoed for the same reason as before.

The Boyer-Lindquist ‘unit’ vector in the θ-direction is thus,

$$\hat{\theta}_{BL} \equiv \frac{\partial}{\partial\theta} = \frac{\partial\eta}{\partial\theta}\cdot\frac{\partial}{\partial\eta} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial}{\partial z} = \frac{\partial\eta}{\partial\theta}\cdot\hat{\eta} + \frac{\partial z}{\partial\theta}\cdot\hat{z}$$
$$= \sqrt{a^2 + r^2}\cos\theta\cdot\hat{\eta} - r\sin\theta\cdot\hat{z} \tag{7}$$

If we interpret $\hat{\eta}, \hat{z}$ as orthogonal unit vectors, the Boyer-Lindquist $\hat{\theta}_{BL}$ therefore does not have unit length. A true unit vector is provided by normalising $\hat{\theta}_{BL}$ as follows,

$$\hat{\theta} \equiv \frac{\hat{\theta}_{BL}}{\left|\hat{\theta}_{BL}\right|} = \frac{\sqrt{a^2 + r^2}\cos\theta\cdot\hat{\eta} - r\sin\theta\cdot\hat{z}}{\rho} \tag{8}$$

The result is that we must divide E_θ^{BL} by ρ to get the correct component for our purposes, hence,

$$E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \qquad \text{and} \qquad E_\theta = -2qa^2 r\sin\theta\cos\theta/\rho^5 \tag{9}$$

3. Plots of the Electric Field Vector

Figures 1 to 9 show various depictions of the electric field. All these plots have $a = 1$.

Figures 1-3 attempt to plot the field by showing both the direction and the magnitude, indicating the latter by the length of the indicative arrows. However, this is unsuccessful because the strength of the singularity at the ring causes all the vectors away from the ring to appear as dots. Removing a region containing the singularity from the plots should help, but in practice produced little improvement (Figure 3).

Figures 4-9 therefore show only the direction but not the magnitude of the electric field. This is sufficient for our purposes. Figures 4, 6, 7 and 8 show the region between r = 0 and r = 1.5. Figure 5 is a close up of the singularity region. These Figures clearly illustrate,

- The field lines emanate from the ring, implying that there is a line-density of positive charge on the ring;
- The field lines terminate on both the upper and lower faces of the disc, implying that there is a surface density of negative charge on both disc surfaces;
- At large enough radii the field lines begin to approximate that of a positive point charge.

The latter feature is clearest in Figure 9, which shows a larger region up to r = 4. At r = 4 the electric field appears to emanate from the origin, i.e., from the centre of the disc (as placing a ruler on the Figure proves).

The answer to the question, “where is the charge” is thus, “both on the ring *and* on the disc, but with opposing signs”. It is clear from the Figures that the charge seen at infinity is less than the charge on the ring due to partial cancellation by the negative charge on the disc surfaces.

What is the ratio of the charge on the ring to that seen at infinity? We answer that next…

4. The Charge on the Ring and the Disc Separately

The algebraic sum of the charges on the ring and the disc equals the charge seen at infinity, which is q. But what are the charges on the ring and the disc separately? To evaluate this we note that Gauss's theorem tells us that the charge on a surface is given in terms of the normal component of the electric field by,

$$\text{Charge} = \oiint \bar{E} \cdot d\bar{A} / 4\pi \tag{10}$$

We are interested in integrating over the surface of the disc, at which the normal component of the electric field is the value of E_r with r = 0, i.e.,

$$E_r(r=0) = -\frac{q}{a^2 \cos^2\theta} \tag{11}$$

To begin with let us naively assume that the area element is just that which would obtain for a Euclidean space, i.e., on r = 0,

$$dA = 2\pi\eta \cdot d\eta = 2\pi a^2 \sin\theta \cos\theta \cdot d\theta \tag{12}$$

Substituting (11) and (12) into (10) gives,

$$\text{Charge} = -\int_0^{\pi/2} \tan\theta \cdot d\theta \tag{13}$$

(noting that the two surfaces of the disc introduces a factor of 2). But the integral in (13) is divergent at the ring, i.e., at $\theta = \pi/2$, since the integral of $\tan\theta$ is $-\log(\cos\theta)$.

We may suspect that this divergence arises because we have used the Euclidean area element, not the correct area element for the Kerr-Newman spacetime geometry. To derive the latter we note that the volume element is given by,

$$dV = \sqrt{-g}\, dr d\theta d\phi \tag{14}$$

where g is the determinant of the covariant metric tensor, as given in (1). This gives,

$$\sqrt{-g} = \rho^2 \sin\theta \tag{15}$$

Now the 'radial thickness'of this volume element can be obtained by setting all differentials other than dr to zero in (1), which gives,

$$\text{Radial thickness} = ds_r = \sqrt{g_{rr}}\, dr = \frac{\rho}{\sqrt{\Delta}} dr \tag{16}$$

Dividing the volume element by the radial thickness gives the area element since $dV = ds_r \cdot dA$. Hence,

$$dA = \sqrt{\Delta} \cdot \rho \sin\theta \cdot d\theta d\phi \tag{17}$$

On r = 0 this becomes,

$$dA = a\sqrt{a^2 + q^2} \cdot \cos\theta \sin\theta \cdot d\theta d\phi \tag{18}$$

Since there are no ϕ dependencies we can consider just the annular area element,

$$dA = 2\pi a\sqrt{a^2 + q^2} \cdot \cos\theta \sin\theta \cdot d\theta \tag{19}$$

But this differs from the Euclidean area element, (12), only by the constant factor $\sqrt{1 + q^2 / a^2}$, and hence the integral for the charge still diverges. The Kerr-Newman geometry has not saved us.

We conclude that the total charge on the disc is $-\infty$ due to the divergent contribution from near the ring singularity.

Since the total charge is finite it follows that the charge on the ring must be $+\infty$. The finite charge seen at infinity is due to a cancellation of two divergent quantities to leave a finite difference, namely *q*.

5. Analytic Expression for the Charge Density

The charge density is the time component of the current vector, $\Im^{\mu}$. The 4-vector current equals the covariant divergence of electromagnetic field tensor,

$$\Im^{\mu} = F^{\mu\nu}{}_{;\nu} \tag{20}$$

The covariant components of the electromagnetic field tensor are given in terms of Boyer-Lindquist coordinates by,

$$F_{rt} = E_r = q\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \tag{21a}$$

$$F_{\theta t} = E_{\theta}^{BL} = -2qa^2 r\sin\theta\cos\theta/\rho^4 \tag{21b}$$

$$F_{\theta\phi} = B_r = 2qar\left(a^2 + r^2\right)\sin\theta\cos\theta/\rho^4 \tag{21c}$$

$$-F_{r\phi} = B_{\theta} = qa\sin^2\theta\left(r^2 - a^2\cos^2\theta\right)/\rho^4 \tag{21d}$$

Using (21a-d) together with the explicit expression, (1), for the metric tensor, it is ‘merely’ a matter of algebra to form the divergence of the contravariant electromagnetic field, as in (20), and hence to find the current. Performing this algebra by hand is a prodigious feat (at which I repeatedly failed). However, it is accomplished very easily using the GR facilities in MAPLE, which confirms that $\Im^{\mu}$ is in fact identically zero.

Consequently, we conclude that the charge density is everywhere zero in the Kerr-Newman solution. There is no electrostatic source.

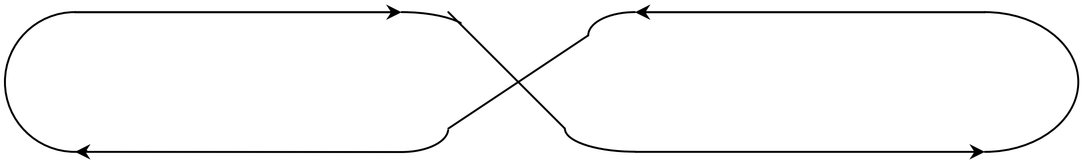
6. Explanation in terms of the Maximally Extended Spacetime

Here we discuss the resolution of the paradox regarding how charge can appear to exist, as judged from the electric force lines at infinity, and yet have no source. The answer is that Figures 1-9 are only half of the complete spacetime, namely that half which has r > 0. There is another region with r < 0 in which the electric field looks identical to Figures 1-9 except that the direction of all the arrows is reversed. Thus, in the r < 0 region, the lines of force *emerge from* the disc, rather than appearing to end on the disc. In other words, the charge density on the disc in the r < 0 region appears to be positive. Similarly, the ring in the r < 0 region is a sink for the field lines, thus appearing to be a negative charge.

The r > 0 and r < 0 regions are joined together at the disc. The upper face of the disc in r > 0 is identified with the lower part of the disc in r < 0, and vice-versa. With this topology we see that, in reality, the field lines do not terminate on the disc at all but simply cross over continuously from the r> 0 region to the r < 0 region or the reverse. There is no charge on the disc because the field lines are continuous and do not terminate or emerge from it. The appearance of a surface charge density was an illusion brought about by confining attention to only part of the whole spacetime.

The same phenomenon occurs at the ring. It appears that field lines emerge from it when considering the r > 0 region in isolation. But in the r < 0 region, the same number of force lines enter the ring. In fact, the lines of force are continuous and merely cross over from the r < 0 region into the r > 0 region.

We need only imbue the spacetime with a particular topology at spatial infinity to complete the job. We identify $r = +\infty$ and $r = -\infty$. The topology of the extended spacetime is thus shown figuratively below,



Conclusion

The spacetime geometry and topology of the Kerr-Newman solution of the Einstein field equations with $m < \sqrt{a^2 + q^2}$ lead to the appearance at infinity of an electric and magnetic source, but actually there is no charge present. Instead the lines of force form a topological loop with no termination points. Is this the true nature of electric charge?

Figure 1:
Ilustrating the difficulty in plotting the magnitudes of the electric field due to the dominance of the ring singularity (MAPLE plot)

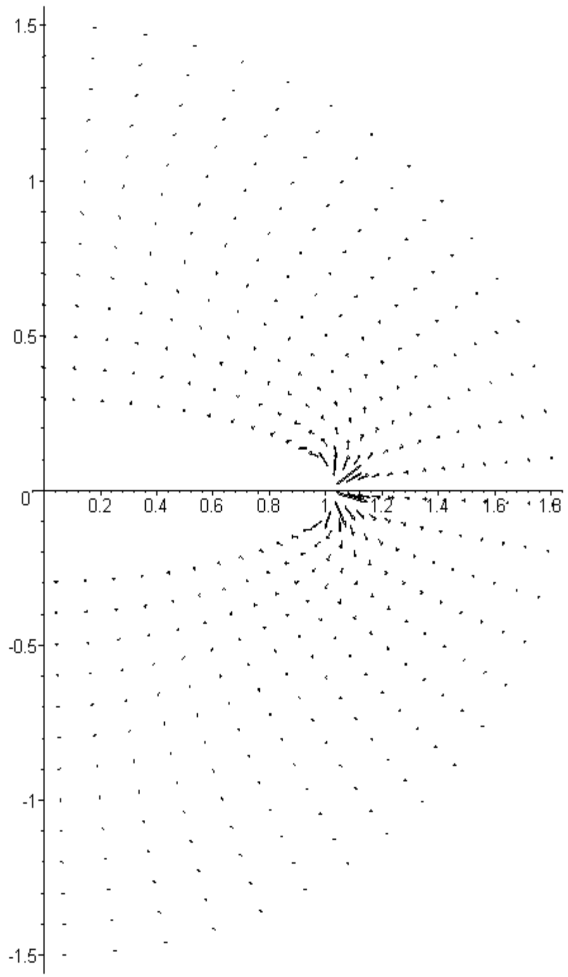


Figure 2:
Ilustrating the difficulty in plotting the magnitudes of the electric field due to the dominance of the ring singularity (MATLAB plot)

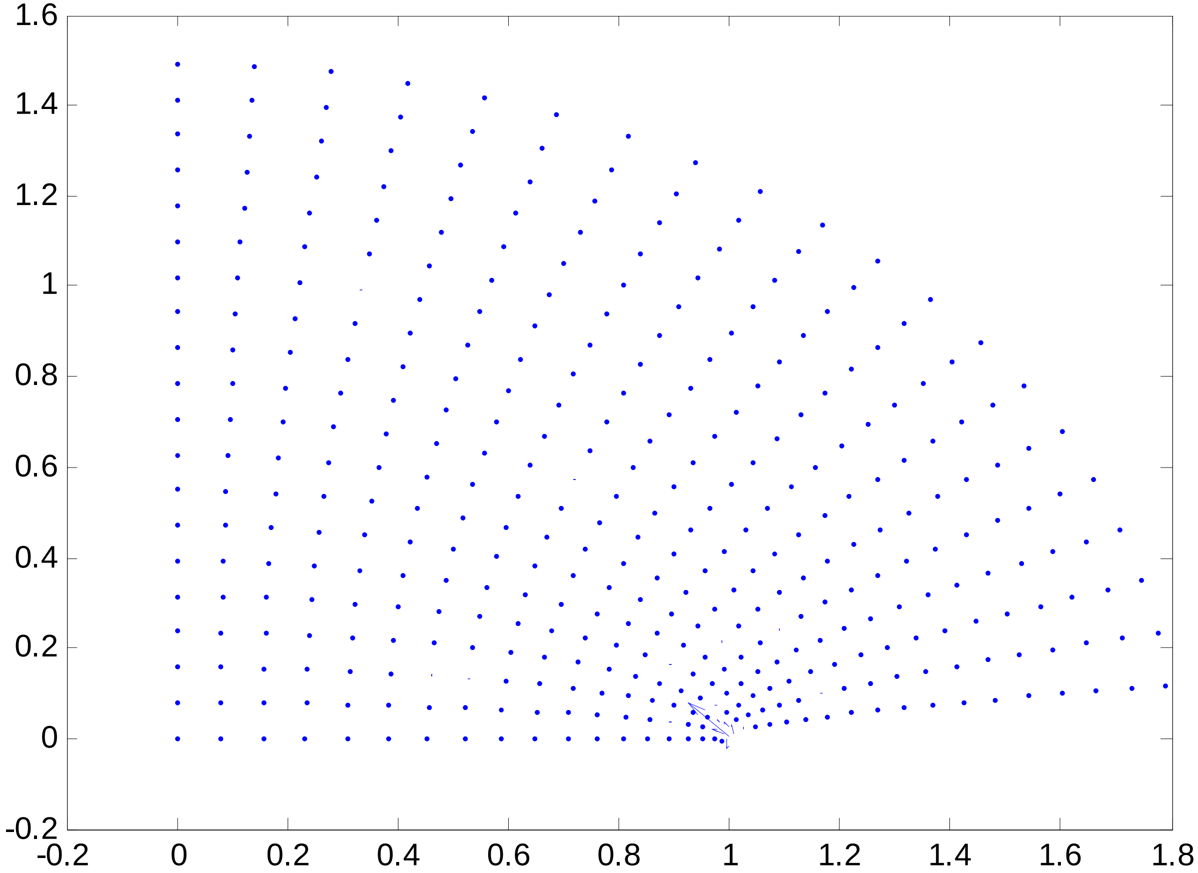


Figure 3:
Ilustrating the difficulty in plotting the magnitudes of the electric field due to the dominance of the ring singularity: the problem persists even when the region near the singularity is removed from the plotted region (MATLAB plot)

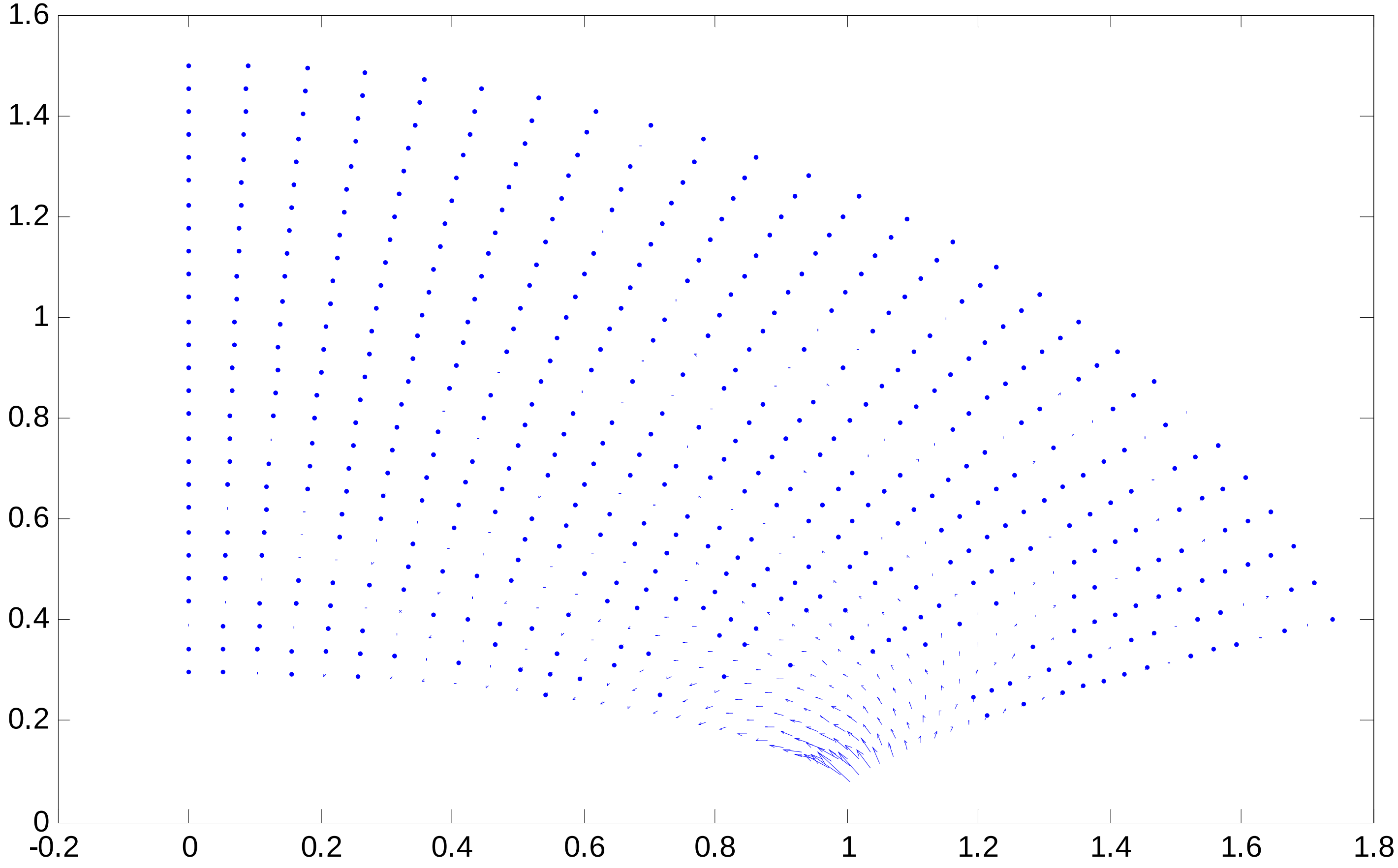


Figure 4: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes) - MAPLE

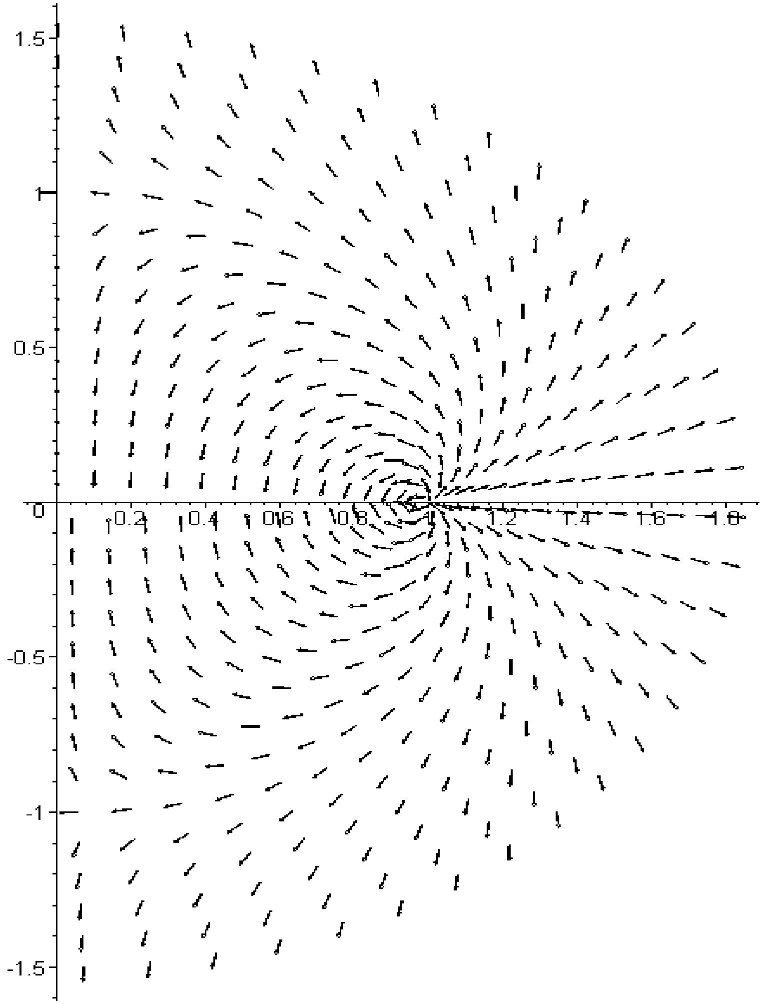


Figure 5: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes): Zoom Near Singularity - MAPLE

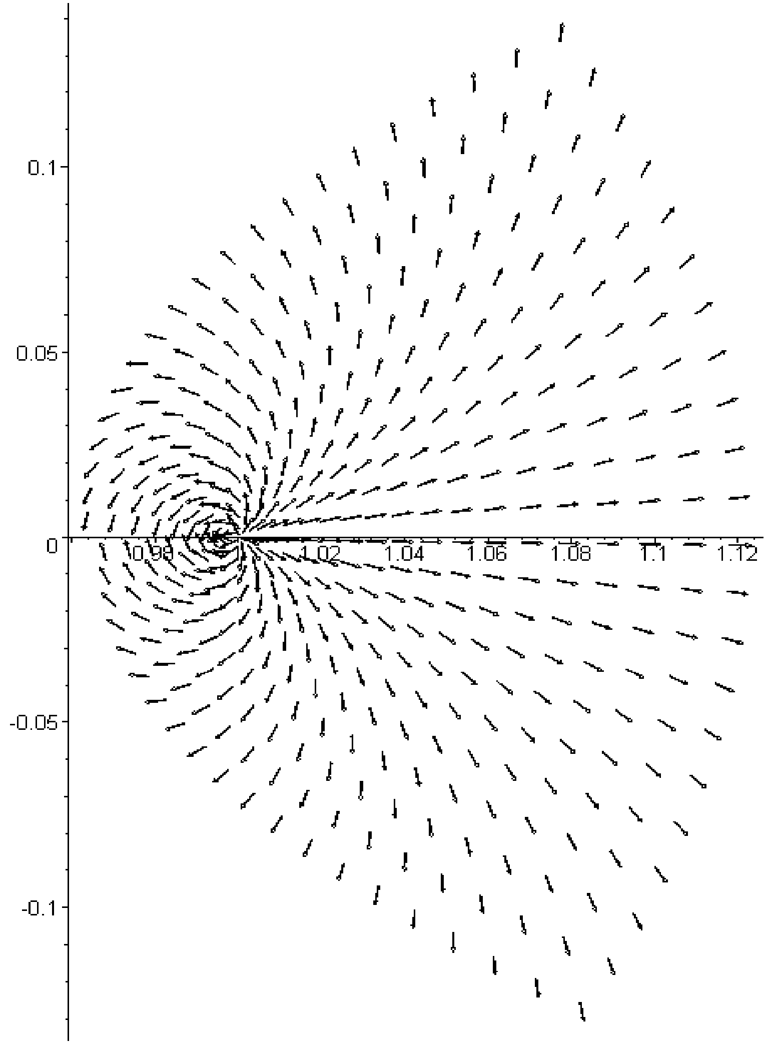


Figure 6: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes) - MATLAB (20 x 20 plot grid)

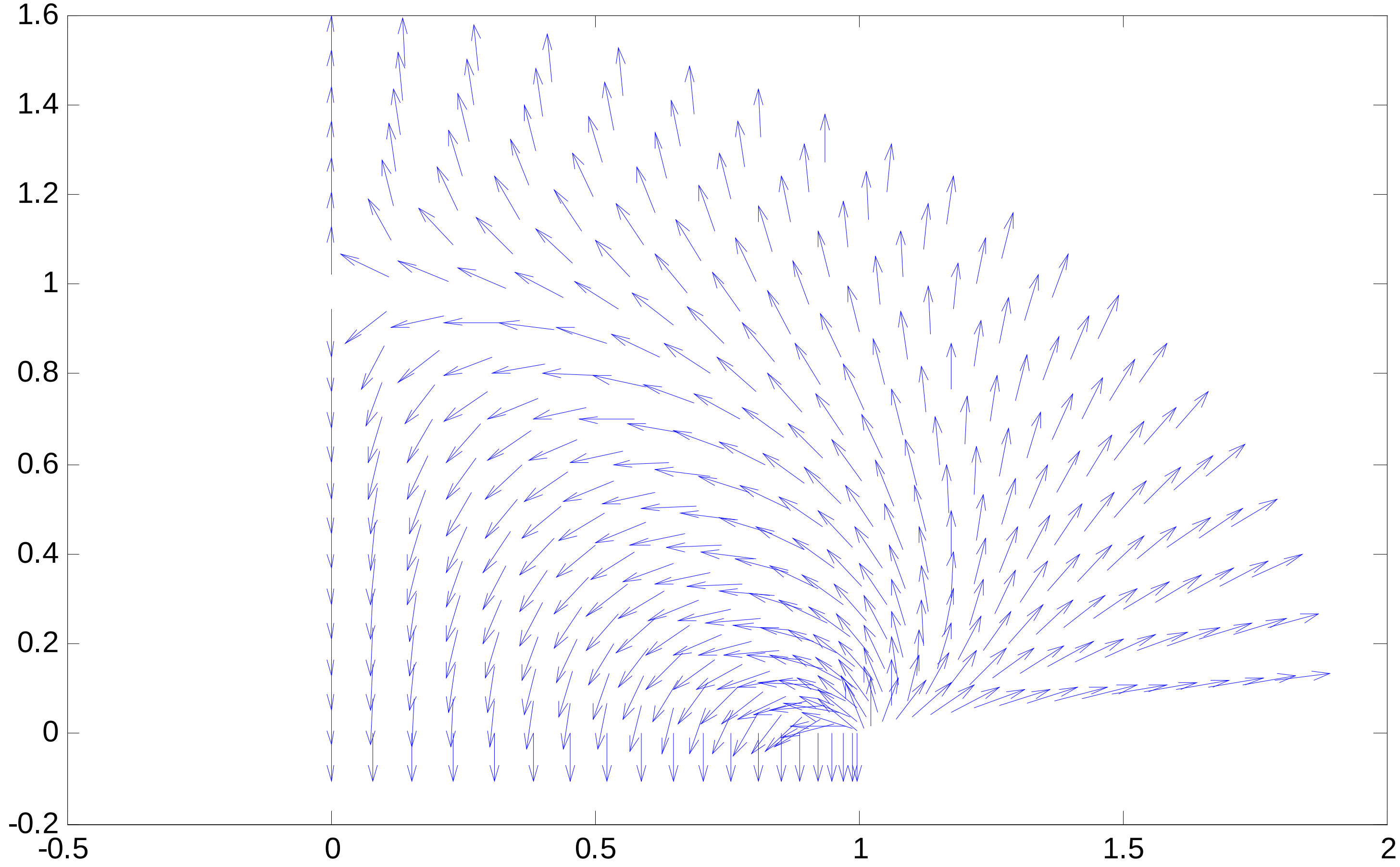


Figure 7: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes) - MATLAB (40 x 40 plot grid)

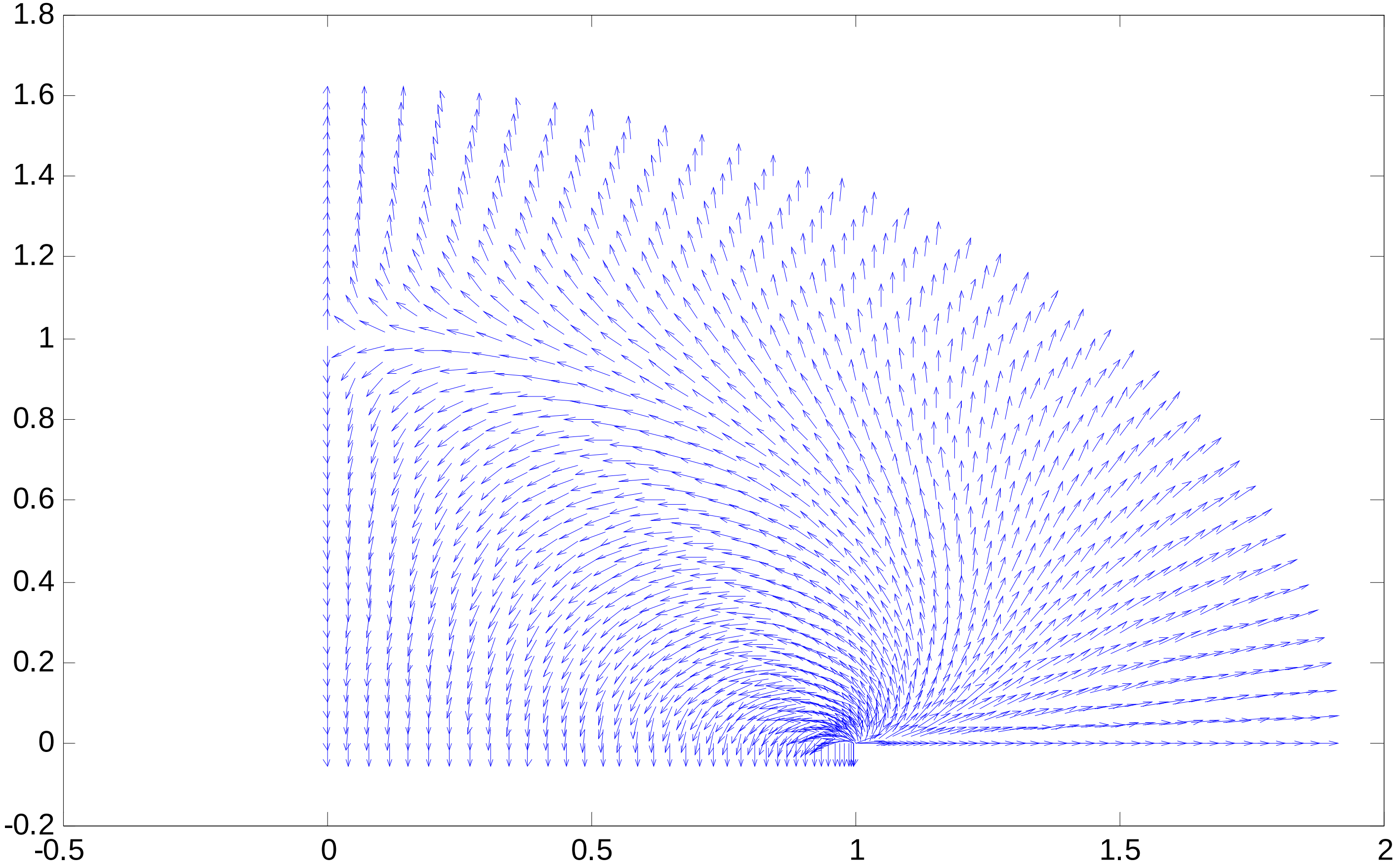


Figure 8: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes) - MATLAB (30 x 30 plot grid)

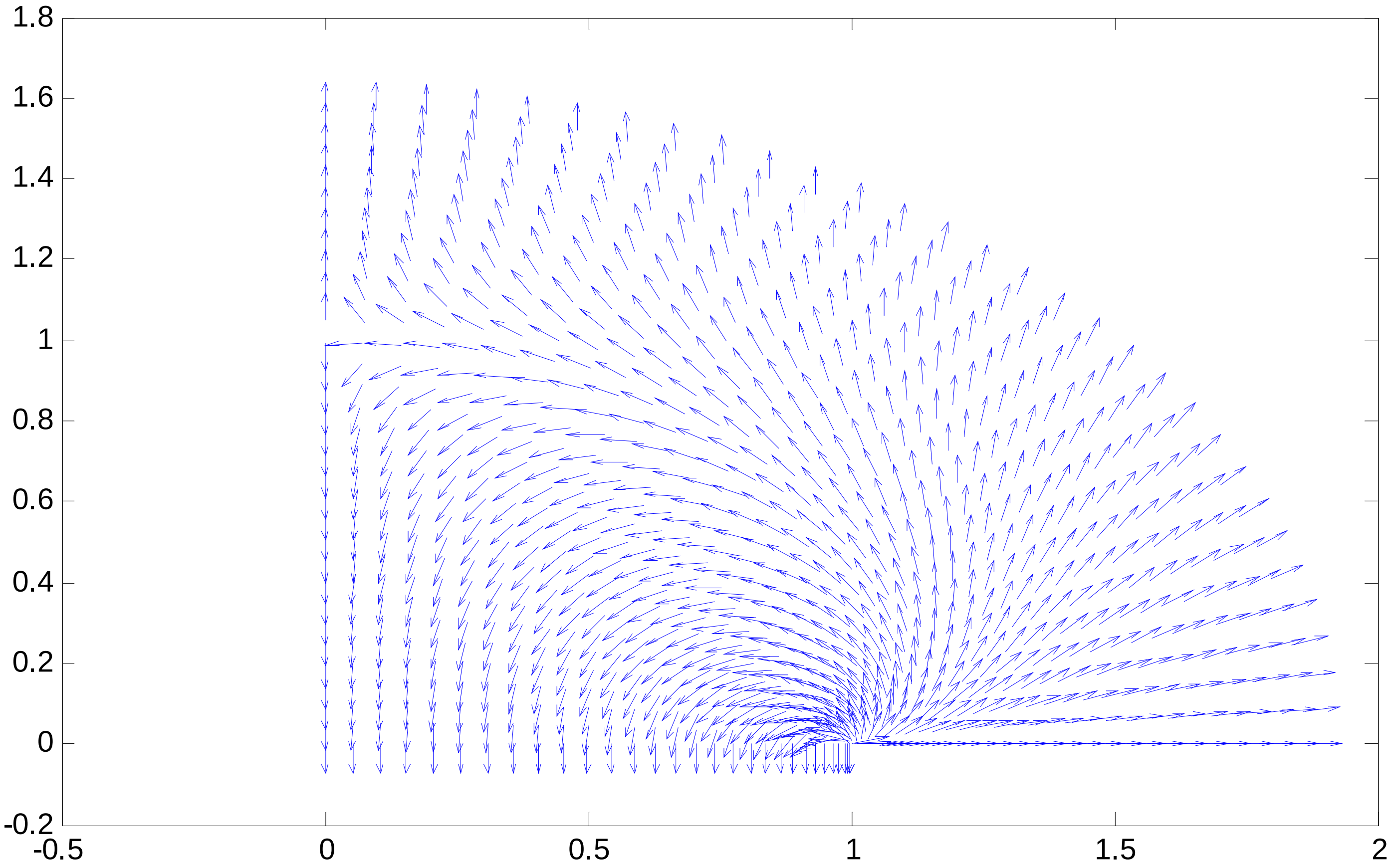


Figure 9: Plot Showing Directions of Electric Field Vector Only (Not Magnitudes) - MATLAB
(20 x 20 plot grid: Larger Region)