Effective Stress Intensity Factor – Energy Release Rate Relationship for an Anisotropic Material

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The linear elastic solution for the crack tip fields in a general orthotropic medium has been solved by Sih and Liebowitz (Ref.1). Knowledge of the stress and displacement fields then leads to the energy release rate, and hence the relation between K and G. Although this relates only to linear elastic behaviour, the resulting equation can be used as the definition of the effective K (i.e. the toughness) after yielding, just as it is in the isotropic case.

Equ.(4.51) of Ref.1 is the required relationship, although expressed in terms of a non-standard definition of K and a general "a-tensor" of elastic constants. The conventional K differs by a factor of $\sqrt{\pi}$. The "a-tensor" for a 2D problem reduces to the following relation between strain and stress components for the general orthotropic medium,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

The crack is taken to lie on the x-axis. Only the Mode I problem is considered here (though Ref.1 addresses all three Modes). The applied tension is in the y-direction. In this case the crack tip fields turn out not to depend upon the constants a_{16} and a_{26}. The remaining constants are more familiar in the following form, for plane stress:-

- $a_{11} = 1/E_1$, where E_1 is the elastic modulus in the x-direction;
- $a_{22} = 1/E_2$, where E_2 is the elastic modulus in the y-direction;
- $a_{66} = 1/\mu$, where μ is the shear modulus in the x-y plane;

There are two Poisson's ratios, depending upon whether the x-strain due a y-stress is considered (ν_1), or vice-versa. But they are not independent, being related by,

- $a_{12} = -\dfrac{\nu_1}{E_1} = -\dfrac{\nu_2}{E_2}$

Provided that the crack is constrained to grow in the self-similar direction (i.e. in the x-direction) the energy release rate is given in terms of the conventional K by:-

Plane stress: $$G = \left(\frac{1}{2E_1E_2}\right)^{1/2}\left[\sqrt{\frac{E_1}{E_2}} + \frac{E_1}{2\mu} - \nu_1\right]^{1/2} K_I^2$$

A similar expression applies in plane strain with the usual adjustments to the constants, i.e., Plane strain: $$G = \left(\frac{1}{2E_1'E_2'}\right)^{1/2}\left[\sqrt{\frac{E_1'}{E_2'}} + \frac{E_1'}{2\mu} - \nu_1'\right]^{1/2} K_I^2$$

where, $E_1' = \dfrac{E_1}{1-\nu_1^2}$, $E_2' = \dfrac{E_2}{1-\nu_2^2}$, $\nu_1' = \dfrac{\nu_1}{1-\nu_1}$ and $\nu_2' = \dfrac{\nu_2}{1-\nu_2}$.

The above expressions are the most general, since the four constants E_1, E_2, μ and ν_1 are independent in general. However, if we make the assumption that the shear modulus can be approximated by,

$$\mu \approx \frac{E_1}{2(1+\nu_1)}$$

then the following G-K relationships hold,

Plane stress: $$G = \frac{\sqrt{1+\sqrt{\frac{E_1}{E_2}}}}{\sqrt{2E_1E_2}} K_I^2$$

Plane strain: $$G = \frac{\sqrt{\left(1-\nu_1^2\right)\left(1-\nu_2^2\right)\left(1+\sqrt{\frac{E_1}{E_2}}\right)}}{\sqrt{2E_1E_2}} K_I^2$$

References

[1] H.Liebowitz (ed), "*Fracture: An Advanced Treatise*", Volume II, Academic Press 1968: Chapter 2, "*Mathematical Theories of Brittle Fracture*", by G.C.Sih and H.Liebowtz - Section IV, "*Rectilinearly Anisotropic Bodies with Cracks*".