

# The Bree Problem with Primary Load Cycling In-Phase with the Secondary Load

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## Abstract

Shakedown and ratcheting behaviour is derived for uniaxial primary membrane and thermal bending stresses both cycling strictly in-phase. The complete analytic solution for perfect plasticity is presented, including the shakedown/ratcheting diagram and the ratchet strains or cyclic plastic strains where relevant. This cyclic loading is shown to be considerably more benign than that of the original Bree problem in which the primary membrane stress does not cycle.

## Keywords

Shakedown, Ratcheting, Bree

## 1. Introduction

The Bree problem, Ref.[1], concerns uniaxial shakedown and ratcheting behaviour under a constant primary membrane stress,  $\sigma_p$ , plus a secondary wall-bending stress which cycles between zero and some maximum elastic value,  $\sigma_t$ . The secondary bending stress is strain controlled and is considered to be due to a uniform through-wall temperature gradient with the bending being fully restrained. Hence a net membrane strain might arise but the total bending strain is constrained to zero. Whilst the application that Bree had in mind in his original analysis, Ref.[1], was fast reactor fuel clad, and hence a cylindrical geometry, he analysed the problem as if for uniaxial stressing. Consequently the problem may be considered as relating to a beam of rectangular section.

For an elastic-perfectly plastic material this problem is analytically tractable, see Ref.[1]. On a plot of  $\sigma_t$  against  $\sigma_p$ , regions can be identified in which the behaviour is qualitatively different, namely, (i)elastic; (ii)elastic cycling after initial plasticity (strict shakedown); (iii)stable plastic cycling without ratcheting; (iv)ratcheting. These regions are displayed on Figure 3 of Ref.[1], denoted E, S, P and R respectively. Moreover, the Bree analysis also provides expressions for the cyclic plastic strains in region P and the ratchet strain per cycle in regions R.

Only a few problems are sufficiently simple to admit such analytic solutions. Consequently analyses such as Bree's provide an important archetype. Indeed, despite being published 45 years ago, the Bree problem is still widely cited as a relevant comparator and as a validation case. In the intended application of the original Bree analysis (fast reactor fuel clad) the internal pressure is retained (though reduced) on shutdown when the thermal stresses are removed. Consequently Bree's simplifying assumption of a constant (non-cycling) primary stress is reasonable for this application. However, in many applications to power plant (e.g., steam pipework and headers or boiler tubes) the primary load is due to steam pressure which is necessarily zero at cold shutdown. Consequently there is an interest in cases where the primary load cycles as well as the secondary load. In the case of steam pipework or boiler tubing it is common for the most significant thermal stresses to be long range system stresses. These are related to the absolute steady operating temperature, as opposed from temperature gradients which would be more relevant for thermal transient conditions. But the high temperatures of superheated steam can arise only at high pressure, and, moreover, such plant is generally operated so as to avoid high pressures when cold. Hence, in these circumstances, there can be a strong correlation between the magnitudes of the cycling primary and secondary stresses. Of course, real boiler start-up and shut-down procedures are complex and will not conform exactly to this idealised picture. Nevertheless it will often be closer to reality to assume for such plant that the primary stress cycles in phase with the secondary stress, than to assume that the primary stress is constant.

Consequently this paper considers uniaxial shakedown behaviour for an elastic-perfectly plastic material under a primary membrane stress,  $\sigma_p$ , and an elastic secondary wall-bending stress,  $\sigma_t$ , both of which cycle in phase between these values and zero. The problem is analytically tractable and it will be found that this loading is considerably more benign as regards ratcheting behaviour than the original Bree loading. For this reason any direct application to plant must demonstrate that the

primary and secondary loading cycles do conform accurately to in-phase behaviour. In practice the utility of the solution to be presented may not be in the context of direct application but rather in providing reason to expect a significant alleviation to result when the primary and secondary loads vary in-phase, even if only approximately. Numerical methods could then be used to determine the actual behaviour of a given component, a recourse which is likely to be necessary for real, geometrically complex, structures in any case.

The literature on shakedown/ratcheting analyses is large and a thorough review is beyond the scope of this paper. Problems which admit analytical solutions are scarce, and hence valuable. More commonly some form of approximate or bounding technique is used (often also accommodating creep strains) for example Refs.[2,3,4]. Application to real structures will generally require numerical methods, either using proprietary finite element codes or using specialist codes specific to the purpose, such as the linear matching method, e.g., Refs.[5,6,7]. Using numerical methods, solutions can be found for cases involving multiaxial stressing and strain hardening, e.g., Refs.[4,8,9]. However many analyses continue to assume a loading which includes a non-cycling primary component, as in the original Bree analysis, e.g., Refs.[10,11]. There are published analyses in which the primary load cycles as well as the secondary load, for example Ref.[12]. In fact the original Bree problem (a pressurised cylinder with a through-wall temperature gradient) has been considered with both the thermal and pressure load cycling, Ref.[13]. However Ref.[13] allows the pressure and thermal loads to vary independently, rather than strictly in-phase as we shall assume here. Ainsworth, Ref.[14], has considered a problem like that of Bree but with a parabolic temperature distribution and cycling both the pressure and the thermal strain (and also including hardening). But the temporal variations of pressure and temperature assumed were not strictly in-phase, and in particular permitted full pressurisation whilst the thermal load was at its minimum, and hence is unlike the problem considered here. An example of a shakedown analysis assuming in-phase variation of pressure and thermal loads, conducted using a finite element method, is Ref.[15]. However this involved a geometrically complex tubesheet/pressure shell connection. Its more general implications, if any, are not clear. As far as the author has been able to discover the analytical solution to the Bree problem with both loads varying strictly in-phase has not previously been presented.

## 2. Formulation of the Problem

A normalised through-wall coordinate system,  $x$ , is defined with  $x = 0$  being midwall, and the surfaces being  $x = \pm 1$ . Uniaxial stressing is assumed, with a linear through-wall temperature distribution where  $\Delta T$  is the temperature difference (surface  $x = +1$  minus surface  $x = -1$ ). All stresses are normalised by the yield stress,  $\sigma_y$ , and elastic-perfectly plastic behaviour is assumed. The elastic thermal stress distribution is thus  $xY$ , where  $Y = -E\alpha\Delta T / 2\sigma_y$ , where  $E, \alpha$  are Young's modulus and the coefficient of thermal expansion respectively. This stress arises from the temperature distribution assuming bending is fully restrained. The primary stress is a uniform (membrane) stress, which, when normalised by the yield stress, is denoted  $X$ . The actual, elastic-plastic, stress is written  $\sigma$ , again normalised by the yield stress.

If  $\varepsilon_p$  is the plastic strain at position  $x$ , then when the loads are acting the total mechanical-plus-thermal strain,  $\varepsilon$ , is given by,

$$\text{(On load)} \quad \varepsilon = \sigma + \varepsilon_p - xY \quad (1)$$

In eqn.(1), and throughout, strains are taken to be normalised by the yield strain,  $\varepsilon_y = \sigma_y / E$ . The stress,  $\sigma$ , and the plastic strain,  $\varepsilon_p$ , will generally vary through-thickness, whereas the parameter  $Y$  is independent of  $x$ . The assumption that bending is fully constrained means that the total strain,  $\varepsilon$ , is also uniform through thickness, i.e., a membrane strain independent of  $x$ , though it may vary from one half-cycle to the next. The stress appearing in eqn.(1) must equilibrate with the applied primary membrane load, so that,

$$\text{(On load)} \quad \int_{-1}^{+1} \sigma \cdot dx = 2X \quad (2)$$

When the loads are removed the (elastic) thermal stress is zero and hence eqn.(1) becomes,

$$\text{(Off load)} \quad \varepsilon = \sigma + \varepsilon_p \quad (3)$$

In this analysis the primary load is assumed to vary in proportion to the thermal load, and hence to be removed at the same time that the thermal load is removed, so that,

$$\text{(Off load)} \quad \int_{-1}^{+1} \sigma \cdot dx = 0 \quad (4)$$

Eqn.(4) is the difference between the present problem and that of Bree, Ref.[1], who adopted a loading such that Eqn.(2) also applied when off load.

The yield criterion is simply  $|\sigma|=1$ . If  $|\sigma|<1$  then the increment of plastic strain at that point is zero during the half-cycle in question. On the first half-cycle, i.e., when load is first applied, any region with  $|\sigma|<1$  therefore has zero plastic strain. Hence eqn.(1) implies that the slope of the stress versus  $x$  plot in such regions must be  $Y$  so that the  $x$ -dependence of the RHS of eqn.(1) cancels.

We consider only the case when the primary membrane stress is tensile,  $X > 0$ . In the general case, assuming there is some yielding (i.e., that  $X + Y > 1$ ) the stress distribution after first loading will be,

[A] **Either**, a region in tensile yield,  $\sigma = 1$ , plus a region with  $|\sigma|<1$  and slope  $Y$ , but no region at compressive yield (Figures 1a,b,c);

[B] **Or**, a region in tensile yield,  $\sigma = 1$ , and a separated region in compressive yield,  $\sigma = -1$ , plus a region with  $|\sigma|<1$  connecting them with slope  $Y$  (Figures 2a,b,c,d).

Note that, without loss of generality, we are assuming positive bending stress at positive  $x$ .

In the original Bree analysis, cases [A] and [B] were found to correspond to distinct ratcheting regions (R1 and R2 in Ref.[1]). The same is found for the modified problem, as well as distinct shakedown and stable plastic cycling regions for the two cases. Like the Bree problem, Ref.[1], the result will be expressed on a diagram of  $Y$  versus  $X$ . Note, however, that unlike the Bree case, the ordinate is *not* the elastic stress range, since, for the present problem, the (normalised) elastic stress range is  $X + Y$  due to both loads cycling in-phase.

The next Section derives the shakedown, ratcheting and stable plastic cycling regions corresponding to case [A]. The complete set of results, including case [B], is then simply stated and summarised.

### 3. Solution for Stresses and Strains

#### 3.1 First Half-Cycle

The solution procedure is the same as that used by Bree, Ref.[1]. It is illustrated here for case [A]. Hence the stress after the first half-cycle is assumed to be as shown in Figures 1a,b,c. Note that the dimension  $a$  may be positive or negative. After the first half-cycle the problem is still identical to that of Bree, Ref.[1]. The maximum compressive stress, at  $x = -1$ , is  $\sigma = -\sigma_1$ , where  $\sigma_1 < 1$  by definition of case [A]. The slope of the elastic part of the stress distribution for  $x < a$  must be  $Y$  so that,

$$\sigma_1 = (1 + a)Y - 1 \quad (5)$$

Equilibrium with the applied primary load, eqn.(2), requires,

$$\sigma_1 = \frac{4(1 - X)}{1 + a} - 1 \quad (6)$$

Equating eqn.(5) and eqn.(6) and solving for  $a$  gives,

$$a = \left\{ 2 \sqrt{\frac{1 - X}{Y}} - 1 \right\} \quad (7)$$

Substituting eqn.(7) into eqn.(5) and using  $\sigma_1 < 1$  implies,

$$Y(1 - X) < 1 \quad (8)$$

Inequality (8) establishes that Case [A] lies below the hyperbola  $Y(1 - X) = 1$  on an  $X, Y$  plot, see Figure 3. (We shall refer to this final diagram, yet to be fully derived, throughout the text to assist the reader).

The plastic strain is zero for  $x < a$  whereas for  $x \geq a$  the distribution follows from eqn.(1) and is given by,

$$(x \geq a) \quad \varepsilon_p^1 = Y(x - a) \quad (9)$$

The superscript on the strain denotes the half-cycle number. The stress and plastic strain distributions are illustrated in Figures 1a,b,c.

### 3.2 2<sup>nd</sup> Half-Cycle

The thermal strains and the primary load have now both been removed and so eqns.(3,4) apply. Eqn.(3) implies that the slope of the stress distribution is equal and opposite to that of  $\varepsilon_p$  at all points, where  $\varepsilon_p$  is the *total* plastic strain accumulated to-date, i.e.,  $\varepsilon_p^{1+2} = \varepsilon_p^1 + \varepsilon_p^2$  where the superscripts denote the half-cycle. Any region with non-zero stress slope must be elastic in the 2<sup>nd</sup> half-cycle and hence have  $\varepsilon_p^2 = 0$ , and hence have equal and opposite slope to  $\varepsilon_p^1$ . From eqn.(9) this establishes that any non-yielding region with  $x > a$  in the 2<sup>nd</sup> half-cycle has slope  $-Y$ . Moreover, parts of the region  $x > a$  can depart from the slope  $-Y$  only if they yield in compression. Three cases arise: (i) no yielding in the 2<sup>nd</sup> half-cycle (Figure 1a); (ii) yielding on both surfaces, in opposite senses, in the 2<sup>nd</sup> half-cycle (Figure 1b); (iii) yielding only in compression in the 2<sup>nd</sup> half-cycle (Figure 1c). These are considered in turn.

### 3.3 No Yielding on 2<sup>nd</sup> Half-Cycle (Figure 1a)

The stresses on the two surfaces are denoted  $\sigma_2$  and  $-\tilde{\sigma}_2$ , where  $0 < \sigma_2 < 1$  and  $0 < \tilde{\sigma}_2 < 1$ . Eqns.(3,4) allow these stresses to be found in terms of the dimension  $a$ ,

$$\sigma_2 = \frac{Y}{4}(1-a)^2 \quad \text{and} \quad \tilde{\sigma}_2 = \frac{Y}{4}(1-a)(3+a) \quad (10)$$

Using eqn.(7) these can be written,

$$\sigma_2 = \left( \sqrt{Y} - \sqrt{1-X} \right)^2 \quad \text{and} \quad \tilde{\sigma}_2 = X + Y - 1 \quad (11)$$

Both these stresses must be less than unity. It can be shown that the second inequality is the more restrictive. Hence it is concluded that the stress distribution is of the form of Figure 1a if,

$$X + Y < 2 \quad (12)$$

The 3<sup>rd</sup> half-cycle will simply involve elastic re-loading back to the first half-cycle stress distribution with no further plasticity. Consequently the region above the elastic line  $X + Y = 1$ , but below the hyperbola  $Y(1-X) = 1$ , eqn.(8), and also below the line  $X + Y = 2$ , denoted S1 in Figure 3, is a region of strict shakedown in which elastic cycling occurs after the initial plasticity of the first half-cycle.

### 3.4 Yielding on Both Surfaces on 2<sup>nd</sup> Half-Cycle (Figure 1b)

The case represented by Figure 1b has yielding on both surfaces in the second half-cycle. A little thought shows that the stress distribution must then be symmetrical, as illustrated by Figure 1b, in order to produce zero net load, eqn.(4). But recall that the region of varying stress,  $-\tilde{a} < x < \tilde{a}$ , must have slope equal and opposite to that of the plastic strain, i.e.,  $-Y$ . This establishes the dimension  $\tilde{a}$  to be,

$$\tilde{a} = \frac{1}{Y} \quad (13)$$

But the slope of the strain curve after the first half-cycle is zero for  $x < a$  so the region of varying stress after the 2<sup>nd</sup> half-cycle in Figure 1b must lie entirely to the right of  $x = a$ . In other words we require,

$$a < -\tilde{a} \quad (14)$$

Using eqns.(7) and (13), the inequality (14) becomes,

$$X > 1 - \frac{1}{4}(Y-1)\left(1 - \frac{1}{Y}\right) \quad (15)$$

Inequality (15) defines the region above the curve cg in Figure 3. Recall that it has already been shown that case [A] refers to points below the hyperbola  $Y(1-X)=1$ , eqn.(8), shown on Figure 3 as curve cd. Consequently, this establishes that Figure 1b applies to the region between the curves cg and cd in Figure 3, marked R1. We may anticipate that this is a ratcheting region, as will now be shown.

In the region  $-\tilde{a} < x < \tilde{a}$  the plastic strains due to the second half-cycle are zero,  $\varepsilon_p^2 = 0$ , so the plastic strains are just those of the first half-cycle, as given by eqn.(9), hence eqn.(3) gives,

$$\varepsilon = \sigma + \varepsilon_p^{1+2} = -\frac{x}{\tilde{a}} + Y(x-a) = -aY \quad (16)$$

Now applying eqn.(3) to the region  $x < -\tilde{a}$  and using eqn.(16) gives,

$$\text{Hence,} \quad \varepsilon_p^{1+2} = -aY - 1 = Y - 1 - 2\sqrt{Y(1-X)} \quad (17)$$

where eqn.(7) has been used to substitute for  $a$ . In a similar manner, considering the region  $x > \tilde{a}$ , eqn.(3) gives the plastic strain there after the second half-cycle to be,

$$\varepsilon_p^{1+2} = -aY + 1 = Y + 1 - 2\sqrt{Y(1-X)} \quad (18)$$

The total plastic strain after the second half-cycle is illustrated Figure 1b. To establish that Figure 1b represents a ratcheting condition it is necessary to consider the third half-cycle. On re-applying both the primary and secondary loads the same stress distribution results as for the first half cycle. To find the plastic strains after the third half-cycle consider firstly the region  $x < a$ . Eqn.(1) gives,

$$\varepsilon = 1 - aY + \varepsilon_p^{1+2+3} \quad (19)$$

But the region  $x < a$  is not yielding during the 3<sup>rd</sup> half-cycle, and this region lies entirely within the region  $x < -\tilde{a}$ , hence  $\varepsilon_p^{1+2+3} = \varepsilon_p^{1+2} = -aY - 1$ , from eqn.(17), and so eqn.(19) gives  $\varepsilon = -2aY$ . This result can be used when considering the region  $x > a$ , for which eqn.(1) then gives,

$$\varepsilon_p^{1+2+3} = -2aY + xY - 1 \quad (20)$$

The total plastic strain after the third half cycle is plotted in Figure 1b. This plastic strain distribution equals that after the first half-cycle but increased uniformly by the following ratchet strain,

$$\text{Ratchet Strain per Cycle} = -(aY + 1) = \left(Y - 1 - 2\sqrt{Y(1-X)}\right) \quad (21)$$

(normalised by  $\varepsilon_y = \sigma_y / E$ ). Note that the positive sign of this ratchet strain is ensured by inequality (15), i.e., by being within region R1 on Figure 3, which is thus proved to be a ratcheting region.

### 3.5 Yielding only in Compression on 2<sup>nd</sup> Half-Cycle (Figure 1c)

In the case of Figure 1c, reverse yielding occurs on the 2<sup>nd</sup> half cycle at  $x=1$  but no yielding occurs at  $x=-1$ . It is immediately clear that this cannot lead to ratcheting since yielding has not occurred at  $x=-1$  on either the 1<sup>st</sup> or the 2<sup>nd</sup> half-cycles (and ratcheting requires the build up of uniform plastic strains across the section). However the occurrence at  $x=1$  of tensile plastic strains in the first half-cycle and compressive yielding in the second half-cycle suggests that Figure 1c will correspond to stable plastic cycling. This will now be established.

Note firstly that because the region  $x < b$  in Figure 1c is elastic during the 2<sup>nd</sup> half-cycle, the slope of the stress distribution must be equal and opposite to that of  $\varepsilon_p^1$  over the whole of the region  $x < b$ , namely  $-Y$  in the region  $a < x < b$ , from eqn.(9), but zero in the region  $x < a$ . This establishes that the boundary between the constant stress region ( $\sigma_2$ ) and the region of reducing stress must be identified with  $x = a$ . The known slope of the stress distribution between  $x = a$  and  $x = b$  gives,

$$\frac{\sigma_2 + 1}{b - a} = Y \quad (22)$$

Imposing the requirement for zero net load, eqn.(4), provides a second relationship involving the unknowns  $b$  and  $\sigma_2$ ,

$$\sigma_2(1+b) - \frac{1}{2}(b-a)(\sigma_2+1) - (1-b) = 0 \quad (23)$$

Eqns.(22) and (23) permit the two quantities  $b$  and  $\sigma_2$  to be found. The solution for  $b$  is,

$$b = \left[ (1+a)^2 + \frac{4}{Y} \right]^{\frac{1}{2}} - 1 = 2\sqrt{\frac{2-X}{Y}} - 1 \quad (24)$$

To establish the nature of the region P1 the plastic strains are required, starting with the second half-cycle. In the region  $a < x < b$  there is no yielding on the 2<sup>nd</sup> half cycle so that the total plastic strain is just that given by the first half cycle, eqn.(9). In region  $x < a$  there has been no yielding on either half-cycle so that eqn.(3) in this region after the 2<sup>nd</sup> half cycle is just  $\varepsilon = \sigma_2$ . Finally, in region  $x > b$ , eqn.(3) gives  $\varepsilon = \sigma_2 = -1 + \varepsilon_p^{1+2}$ , hence,

$$\varepsilon_p^{1+2} = 1 + \sigma_2 \quad (25)$$

Note that the plastic strain given by eqn.(9) at  $x=b$  is consistent with eqn.(25). This establishes that the plastic strain after the 2<sup>nd</sup> half-cycle differs from that after the first half-cycle only in region  $x > b$ , as shown in Figure 1c.

Now consider the 3<sup>rd</sup> half-cycle. The first half-cycle stress distribution is regained. There continues to be zero plastic strain in the region  $x < a$  so that eqn.(1) gives  $\varepsilon = 1 + (x-a)Y - xY = 1 - aY$ . In the region  $x > a$  eqn.(1) therefore gives,  $\varepsilon = 1 - aY = 1 + \varepsilon_p^{1+2+3} - xY$  and hence  $\varepsilon_p^{1+2+3} = (x-a)Y$ , identical to eqn.(9). So the plastic strain after the 3<sup>rd</sup> half-cycle returns to exactly the same strain, at all points of the section, as on the first half-cycle – as illustrated in Figure 1c. However, the plastic strain does cycle between distinct values in the region  $x > b$ , the cyclic plastic strain at the outer fibre,  $x=1$ , being,

$$\text{Cyclic Plastic Strain} = (1-b)Y \quad (26)$$

(normalised by  $\varepsilon_y = \sigma_y / E$ ). This establishes that Figure 1c, corresponding to region P1 on Figure 3, is indeed a region of stable plastic cycling, the extent of the cyclic plastic zone being  $x > b$  where  $b$  is given by eqn.(24).

The lower limit of P1, where it borders the strict shakedown region S1, is found from the condition  $b \rightarrow 1$  so that Figure 1c reduces to Figure 1a. Using eqn.(24) this is seen to correspond to the line  $X + Y = 2$ , consistent with the definition of the adjoining region S1 via inequality (12).

### 3.6 The Complete Solution

Sections 3.1 to 3.5 have shown how the solution is derived for case [A], when there is no compressive yielding on first loading. The solution for case [B], when there is yielding in both tension and compression on the first half cycle, follows in like manner. The stress and plastic strain distributions for case [B] are illustrated in Figures 2a,b,c,d and correspond to regions R2, P2, P3 and S2 in Figure 3 respectively. Referring to Figure 3, the final result is,

- Region E never yields;
- Regions S1, S2 produce strict shakedown to elastic cycling after the first half-cycle;
- Regions P1, P2, P3 produce stable plastic cycling;
- Regions R1, R2 produce ratcheting;

The equations for the various boundary curves are,

$ak$	$X + Y = 1$	$cg$	$1 - X = (Y - 1)(1 - 1/Y) / 4$
$hbg$	$X + Y = 2$	$hc$	$Y(2 - X) = 4$
$abcd$	$Y(1 - X) = 1$	$fc$	$XY = 2$

Note that, despite the differing functional form of the boundary curves, the curves  $abcd$ ,  $cg$ ,  $hc$  and  $fc$  all meet at the same point,  $c$ , which is  $X = 0.667$ ,  $Y = 3$ .

The shakedown diagram for the present problem is compared with that for the original Bree problem, Ref.[1], in Figure 4.

Formulae for salient dimensions, stresses and plastic strains are summarised in Table 1, of which the ratchet strains (regions R1 and R2) and cyclic plastic strains (regions P1, P2 and P3) are likely to be of greatest interest.

#### 4. Discussion

Design codes commonly employ a criterion based upon the total linearised elastic stress range, for example  $\Delta(P + Q)$  in ASME, Ref.[16], or R5, Ref.[17], notation, this being the sum of the primary and secondary membrane plus bending stress ranges. The acceptable limit of this quantity often approximates to twice the lower bound yield strength. For example, in ASME, Ref.[16], the limit is  $3S_m$  which, if time-independent properties dominate, may be  $2\sigma_y$ . In contrast, the original Bree problem has the disconcerting feature that as  $X \rightarrow 1$  ratcheting can occur even for a vanishingly small secondary stress range. The significance of this is discussed below.

For the modified problem, even as  $X \rightarrow 1$ , ratcheting cannot occur unless  $Y > 1$  and this corresponds to a stress range of at least twice yield. Moreover, so long as  $X < 0.875$  ratcheting cannot occur unless  $Y > 2$ , corresponding to an elastic stress range of 2.875 times yield. Assuming the typical design code primary membrane stress limit is obeyed, i.e.,  $X < 0.67$ , then ratcheting will not occur for the modified problem provided that the secondary stress does not exceed  $3\sigma_y$ . Hence for situations approximating to the modified problem, and assuming the primary membrane stress limit is obeyed, the “twice yield” criterion is very conservative as regards avoiding ratcheting. For example, if  $X = 0.67$  and  $Y = 3$  ratcheting is just avoided but the elastic stress range is  $3.67\sigma_y$ , very substantially greater than the typical “twice yield” limit required in design codes.

In contrast, for situations approximating to the original Bree problem, and assuming  $X = 0.67$ , ratcheting occurs for  $Y > 1.33$  and hence for an elastic stress range of only  $1.33\sigma_y$ . There is therefore an apparent conflict between the Bree diagram, Figure 4, and criteria commonly used in structural codes to avoid ratcheting. However this conflict is probably more apparent than real. Even supposing that the loading is indeed that of the original Bree problem, with thermal stresses cycling during operation at steady pressure, there will generally also be a cold shutdown condition in which both loads vanish. The assessor would therefore take the elastic stress range to be  $X + Y$  for the shakedown assessment. In the limiting case discussed above this would give a stress range of  $0.67 + 1.33 = 2.0$  and hence coincides with the code limit. The sum  $X + Y$  would only drop below 2.0 on the ratchet boundary if  $X > 0.67$  were allowed, but this is prohibited by the primary stress limits employed by most codes. (There could be some residual concern if  $X$  were strictly non-varying, such as a deadweight load).

The moral is that the primary stress limits are significant in ensuring avoidance of ratcheting, as well as the stress range limit. If  $X$  were allowed to encroach on unity, then, for a structure loaded as in the original Bree problem, there could truly be a great sensitivity to small secondary stress variations. The importance of the primary stress limits in protecting against ratcheting is a particular illustration of the broader principle that design codes provide protection only through the totality of their stipulations, and hence that “cherry picking” different code requirements should be avoided.

It should be recalled that both the present analysis and that of Bree, Ref.[1], assume perfect plasticity. This, of course, is to make the problem analytically tractable. Structural codes can include allowance for the effects of strain hardening on shakedown. For example the R5 procedure, Ref.[17], uses a shakedown factor,  $K_S$ , so that the “twice yield” limit is refined as  $(K_S \sigma_y)_{hot} + (K_S \sigma_y)_{cold}$ . This is an experimentally defined factor which is temperature and material dependent and allows for cyclic hardening or cyclic softening. Whilst hardening effects will generally be beneficial, materials which exhibit cyclic softening will be more prone to ratcheting. ASME incorporates allowance for this in the definition of  $S_m$ . For example, in the context of ASME III-NB (Class 1 Components), Ref.[18], Section 6.5.5 states, “The  $P + Q$  limit is  $3S_m$ , not  $2\sigma_y$ .....For perhaps all of the materials that produce cyclic softening,  $S_m$  is based upon one-third of the UTS rather than two-thirds yield

strength. Consequently, the  $P + Q$  limit is less than  $2\sigma_y$ , and there is an allowance for some cyclic softening”.

## 5. Conclusions

The complete solution has been found for the Bree problem modified such that the primary membrane stress cycles in-phase with the secondary bending stress. The shakedown diagram, Figure 3, identifies the regions corresponding to strict shakedown, stable plastic cycling and ratcheting, each region corresponding to qualitatively distinct distributions of stress and plastic strain, Figures 1a-c and 2a-d.

Algebraic expressions for the cyclic plastic strain and the ratchet strain per cycle have been determined and are summarised in Table 1, together with various salient dimensions and stresses.

The simple criterion that the elastic stress range be less than twice yield ( $X + Y < 2$ ) is the necessary and sufficient condition for elastic cycling after the initial loading. This is noteworthy because, for the original Bree problem, an elastic stress range of twice yield does *not* result in shakedown if the primary membrane stress exceeds half yield.

The shakedown diagram is compared with that of the original Bree problem in Figure 4. The present problem, with the primary stress cycling strictly in-phase with the secondary stress, is substantially more resistant to ratcheting. At a given primary stress, the ratchet boundary for the modified problem lies at a secondary stress at least  $\sigma_y$  higher than for the Bree problem. This demonstrates that, for a given maximum total load, the occurrence of ratcheting is strongly dependent upon which loads cycle and their relative phase.

Criteria which limit the elastic stress range to roughly twice yield are poor indicators of ratcheting, being potentially either non-conservative or overly conservative depending upon which loads cycle and their relative phase. Engineers should be aware that structural codes which employ roughly “twice yield” criteria are also reliant upon satisfaction of the primary stress limits as an integral part of the shakedown assessment in order to ameliorate any potential non-conservatism. In more favourable circumstances, such as that analysed here, the code criteria can be substantially conservative.

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**Table 1: Salient Dimensions, Stresses and Plastic Strains**

$\varepsilon_R, \varepsilon_L$  are the plastic strains on first application of load on the right and left surfaces respectively. They are zero if not specified.  $\Delta\varepsilon_R, \Delta\varepsilon_L$  are the cyclic plastic strains on the right and left surfaces respectively, defined for regions P1, P2 and P3 only.  $\varepsilon_{ratchet}$  is the uniform ratchet strain, non-zero only for regions R1 and R2.

Region	Figure	Salient Dimensions, Stresses and Plastic Strains		
S1	1a	$a = 2\sqrt{\frac{1-X}{Y}} - 1$ $\sigma_1 = 2\sqrt{Y(1-X)} - 1$	$\varepsilon_R = 2(Y - \sqrt{Y(1-X)})$ $\sigma_2 = (\sqrt{Y} - \sqrt{1-X})^2$	$\tilde{\sigma}_2 = X + Y - 1$
R1	1b	$a = 2\sqrt{\frac{1-X}{Y}} - 1$ $\varepsilon_R = 2(Y - \sqrt{Y(1-X)})$	$\sigma_1 = 2\sqrt{Y(1-X)} - 1$ $\varepsilon_{ratchet} = (Y - 1 - 2\sqrt{Y(1-X)})$	$\tilde{a} = \frac{1}{Y}$
P1	1c	$a = 2\sqrt{\frac{1-X}{Y}} - 1$ $\sigma_2 = 2(\sqrt{Y(2-X)} - \sqrt{Y(1-X)}) - 1$	$b = 2\sqrt{\frac{2-X}{Y}} - 1$ $\varepsilon_R = 2(Y - \sqrt{Y(1-X)})$ $\Delta\varepsilon_R = 2(Y - \sqrt{Y(2-X)})$	
R2	2a	$a = -X + \frac{1}{Y}$ $\varepsilon_R = XY + Y - 1$	$b = -X - \frac{1}{Y}$ $\varepsilon_L = XY - Y + 1$	$\tilde{a} = \frac{1}{Y}$ $\varepsilon_{ratchet} = XY - 2$
P2	2b	$a = -X + \frac{1}{Y}$ $\varepsilon_R = XY + Y - 1$ $\Delta\varepsilon_R = \frac{XY}{2} + Y - 2$	$b = -X - \frac{1}{Y}$ $\varepsilon_L = XY - Y + 1$ $\Delta\varepsilon_L = -\frac{XY}{2} + Y - 2$	$c = -\frac{2}{Y} - \frac{X}{2}$ $\sigma_0 = \frac{XY}{2}$ $c' = \frac{2}{Y} - \frac{X}{2}$
P3	2c	$a = -X + \frac{1}{Y}$ $\sigma_2 = 2\sqrt{Y(2-X)} - 3$ $\varepsilon_L = XY - Y + 1$	$b = -X - \frac{1}{Y}$ $\sigma_0 = 2\sqrt{Y(2-X)} - Y + XY - 2$ $\varepsilon_R = XY + Y - 1$	$c' = 2\sqrt{\frac{2-X}{Y}} - 1$ $\Delta\varepsilon_R = 2[Y - \sqrt{Y(2-X)}]$
S2	2d	$a = -X + \frac{1}{Y}$ $\sigma_0 = X(Y - 1)$ $\varepsilon_L = XY - Y + 1$	$b = -X - \frac{1}{Y}$ $\sigma_2 = \tilde{\sigma}_2 = X + Y - 1$ $\varepsilon_R = XY + Y - 1$	

Figure 1a: No Compressive Yield on First Loading, Region S1

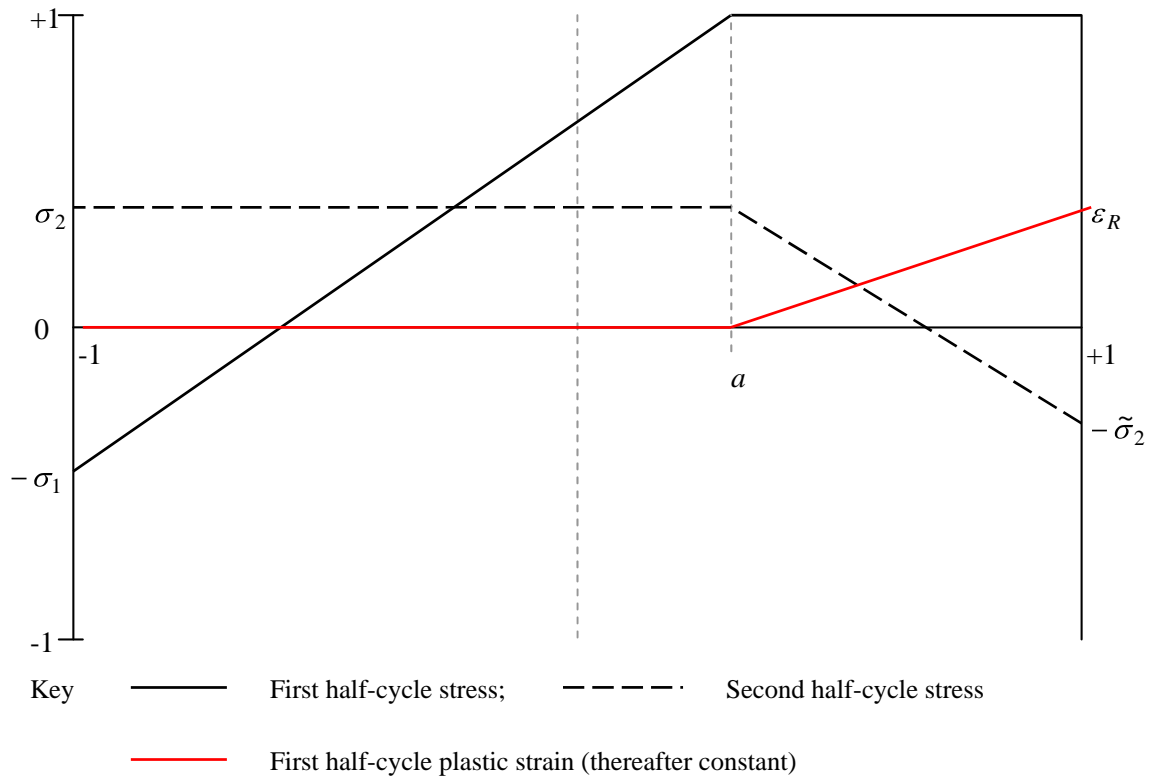
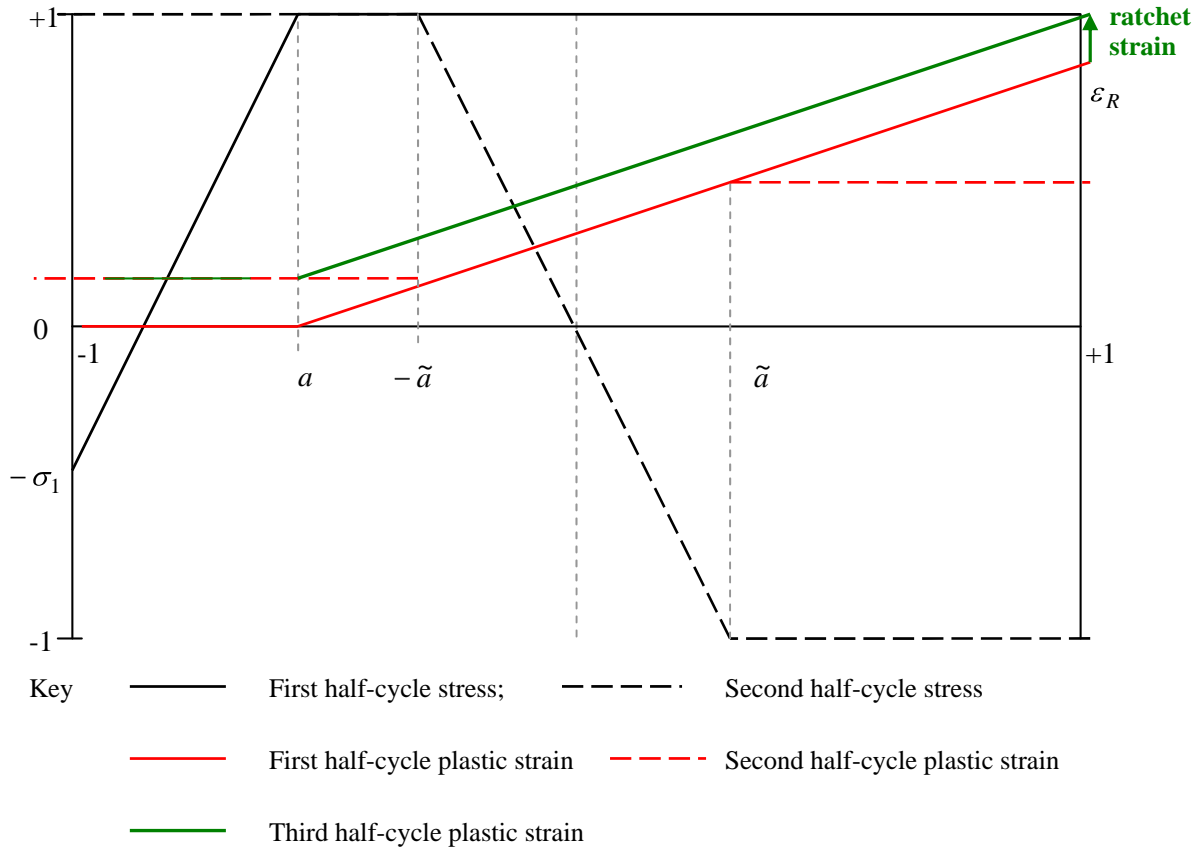


Figure 1b: No Compressive Yield on First Loading, Region R1



(Subsequent half-cycle strains not shown but obtained from the above by displacing upwards by the ratchet strain per complete cycle)

Figure 1c: No Compressive Yield on First Loading, Region P1

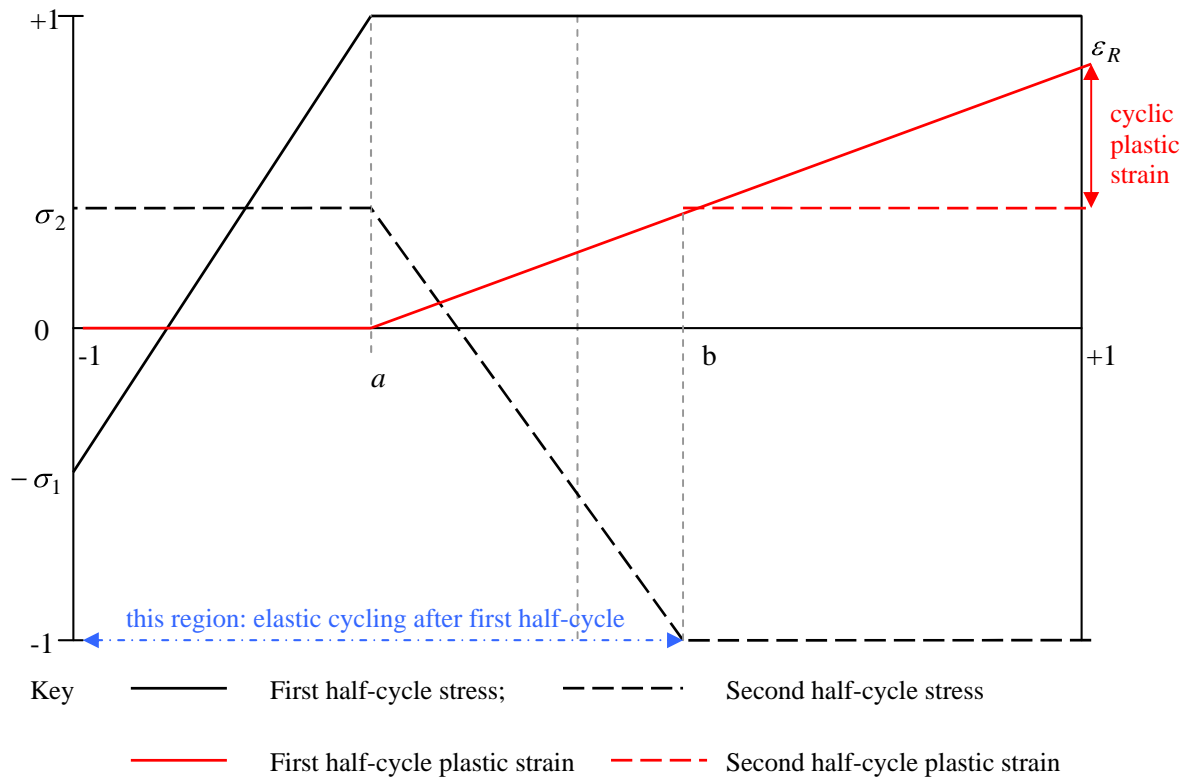
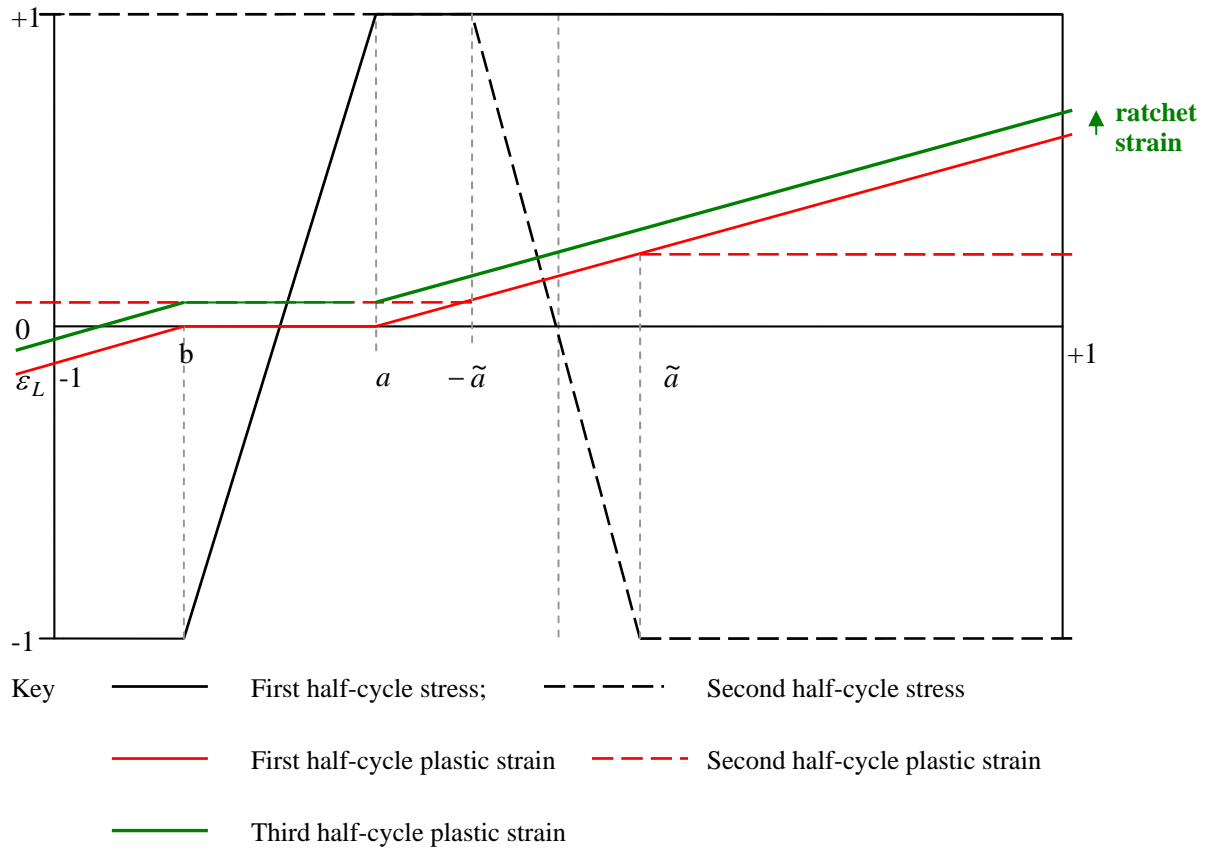


Figure 2a: Compressive Yield on First Loading, Region R2



(Subsequent half-cycle strains not shown but obtained from the above by displacing upwards by the ratchet strain per complete cycle)

Figure 2b: Compressive Yield on First Loading, Region P2

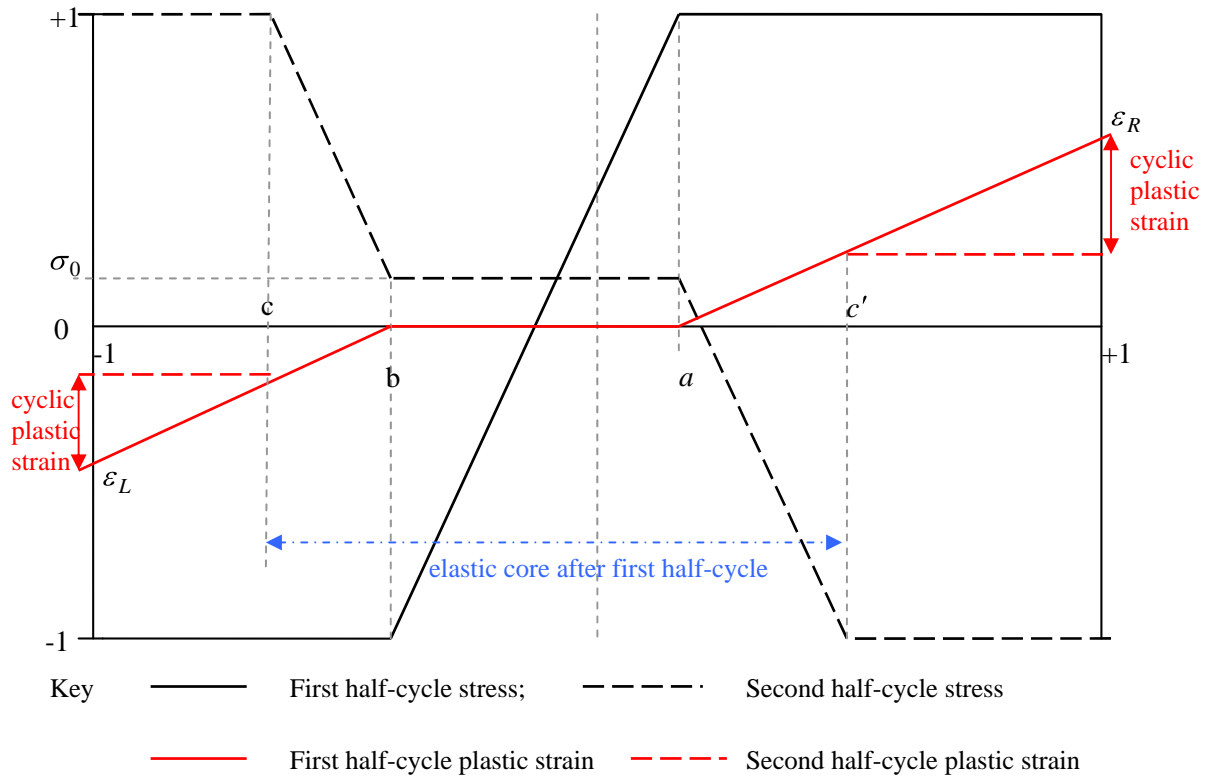
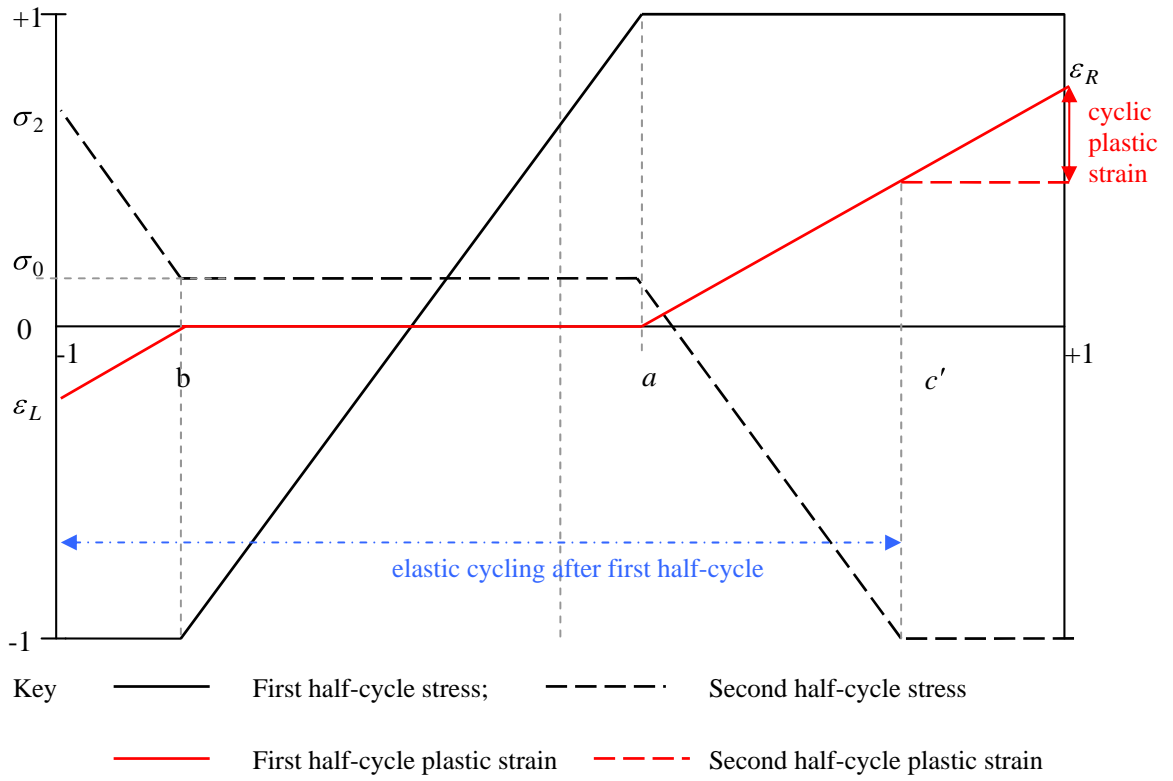
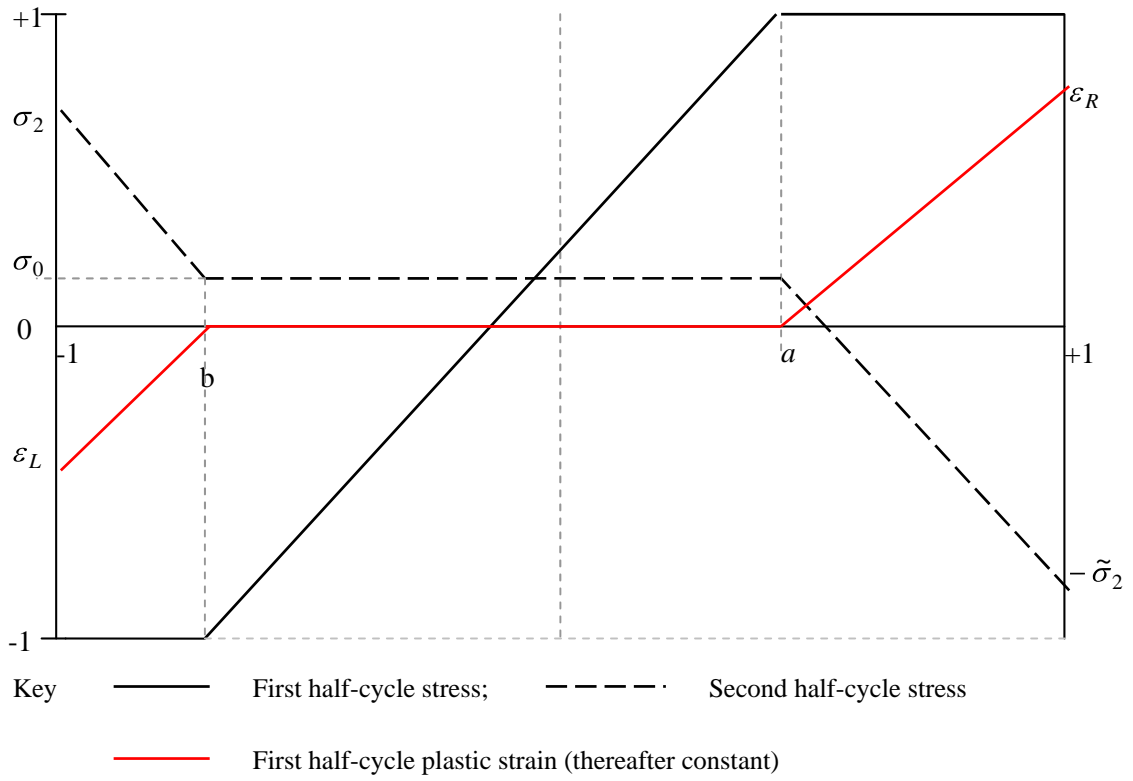


Figure 2c: Compressive Yield on First Loading, Region P3

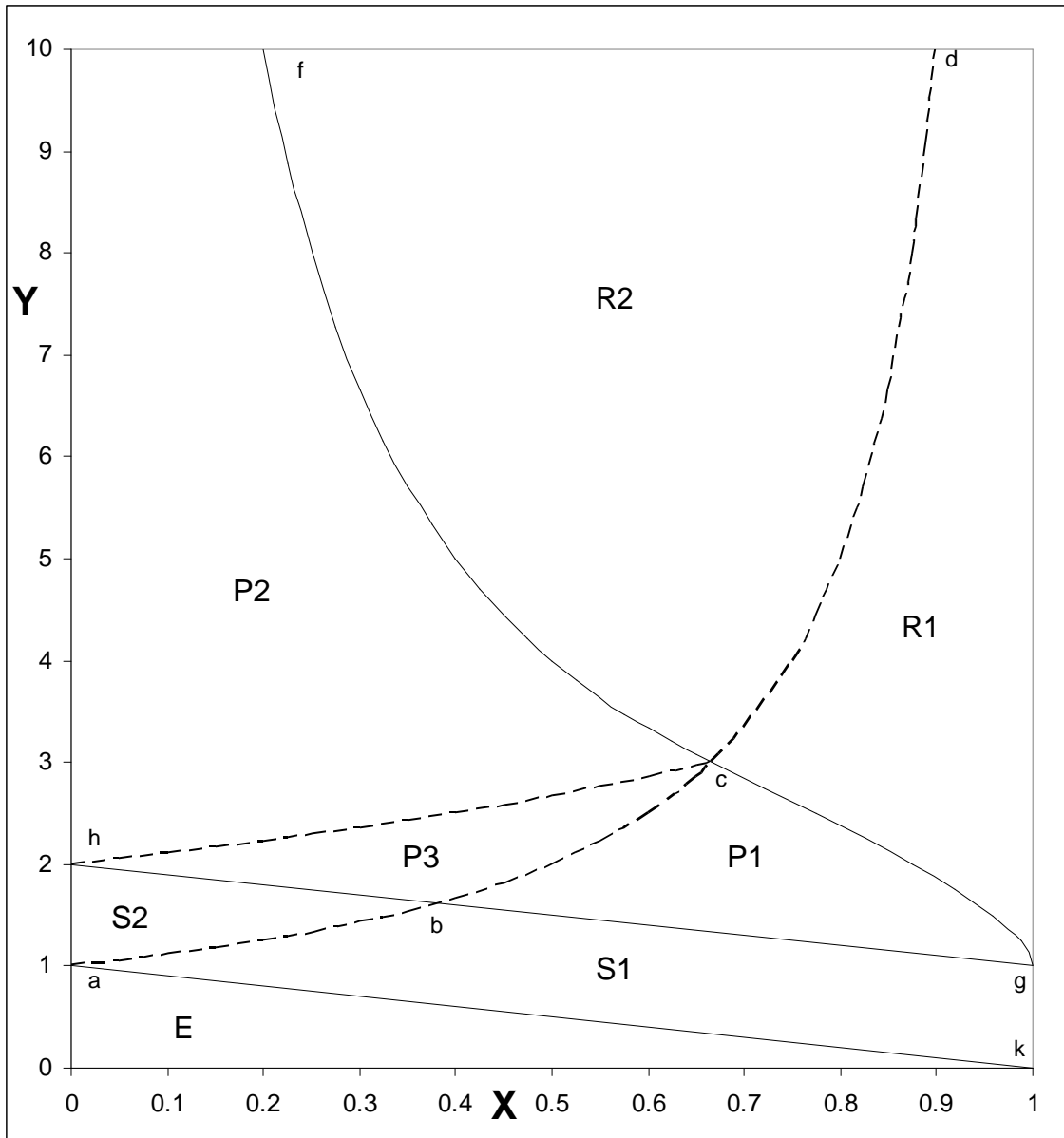




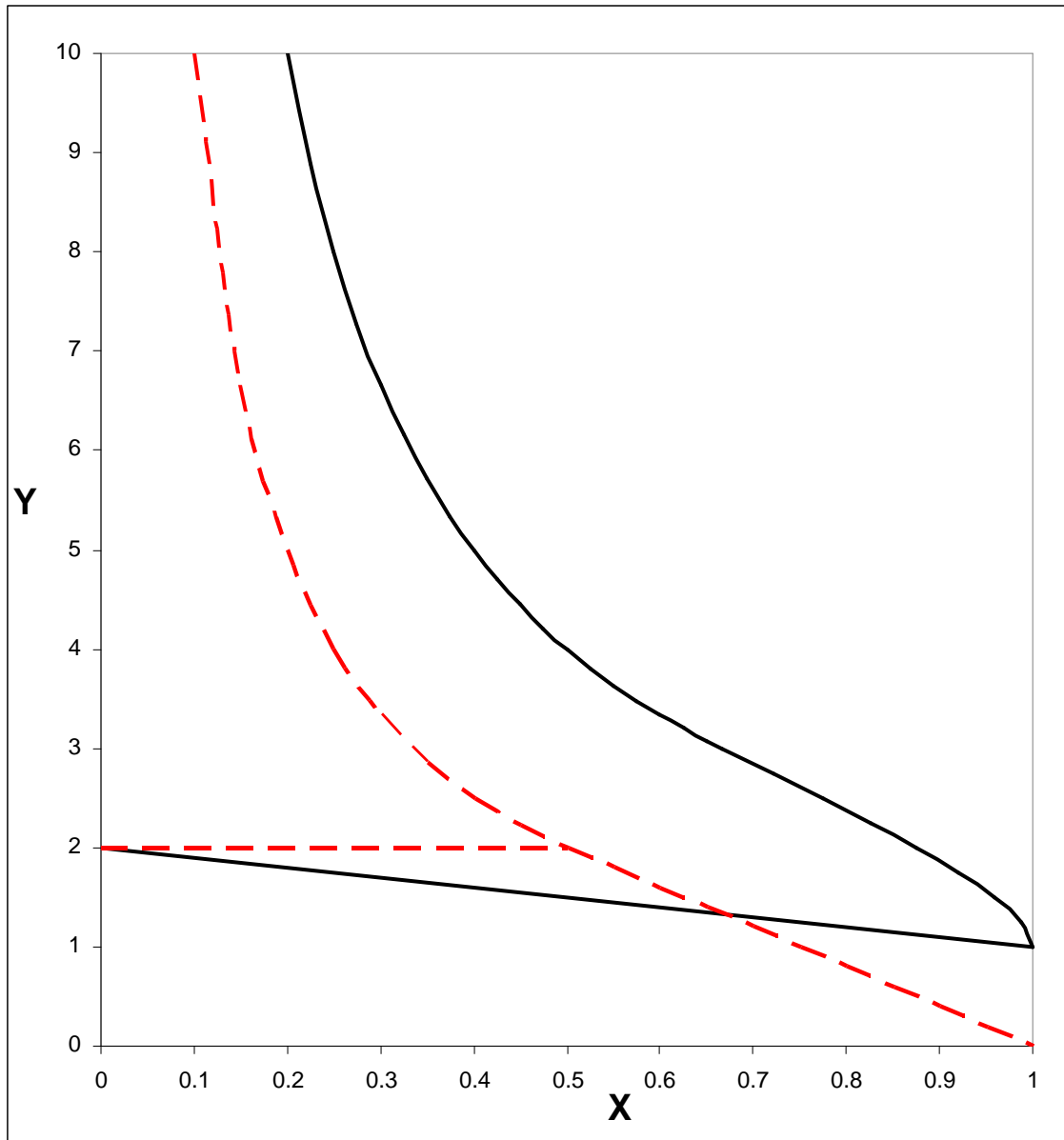
**Figure 2d: Compressive Yield on First Loading, Region S2**



**Figure 3** The Shakedown Diagram (See text for definition of regions)



**Figure 4: Shakedown and ratcheting regions for in-phase cycling primary load compared with those of the original Bree problem (non-cycling primary load).** Continuous black lines are the present work (in-phase cycling primary load); Dashed red lines are the original Bree case, Ref.[1]. For both problems, the upper region is ratcheting, the lower region is elastic or strict shakedown to elastic cycling, and the middle region is stable plastic cycling.



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