

How Long Can A Pencil Remain Balanced On Its Tip?

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1. Introduction

Imagine that a pencil has been successfully balanced on its tip in the vertical position. Imagine also that it is entirely free from disturbances. Thus, all vibrations are rigorously excluded from the table on which the pencil stands. Likewise, not the slightest breath of air is permitted to disturb it, not even the lowest amplitude sound wave. Imagine also that the pencil's tip is sharp and that an infinitesimal disturbance would unbalance it. How long will the pencil remain upright? From the point of view of classical physics this problem is indeterminate. The attempt to play off the absence of disturbance against the pencil's infinite instability merely results in a comparison of one unattainable precondition against another. In the absence of additional information, it is not possible to decide which idealised precondition is the more imperfect, and hence it is not possible to decide whether the pencil will fall. In other words, the question is ill-posed. Curiously, however, this conclusion applies only when the question is considered from the perspective of classical physics.

In contrast, when interpreted as a quantum mechanical question, the question has a well defined answer. The pencil falls. Moreover, the greatest time for which the pencil can remain balanced can be calculated. (More precisely, the probability that the pencil has fallen can be calculated for any given time t). The reason for the difference between the classical and quantum interpretations of the problem lies in the unachievability of the specified conditions in the latter case. A precise, and lasting, localisation of the pencil to the vertical is not consistent with quantum mechanical uncertainty. Preparation of the pencil in a state of definite position (i.e. vertical) leads to divergent uncertainty in its momentum, hence velocity, and thus it falls all the quicker. [We may also remark that the requirement that the pencil be undisturbed is also unachievable. The turbulent quantum mechanical vacuum imparts its zero-point motions to the pencil, and these are irradicable]. Thus, the quantum mechanical problem replaces the ideal conditions with the best approximation that is, in principle, achievable whilst respecting the precepts of quantum physics.

A solution to the problem has been given by Laughlin ("A Different Universe", Note 12 to Chapter 5), albeit described differently. [Laughlin considers a bowling ball in unstable equilibrium atop a convex hill]. The time taken to fall is given by Laughlin as,

$$\text{time to fall} = \sqrt{L/(32g)} \cdot \log[8m^2L^3g/\hbar^2] \quad (1)$$

where, L = the height of the hill (or the height of the centre of gravity of the pencil), g is the acceleration due to gravity, m is the mass of the ball (or pencil), and \hbar is Planck's reduced constant as usual (and \log is natural). Laughlin uses $m = 7.3$ kg and $L = 1$ metre, giving a maximum balancing time of about 9 seconds. [With these data we get $\sqrt{L/(32g)} = 0.056$ seconds, and the log term is 164.8.]. **But Equ.(1) appears to be a factor of $\sqrt{8}$ too small – see Section 4.**

Note that Equ.(1) is very insensitive to the mass. Thus, for $L = 1$ metre, changing the mass from 1 tonne to 1 gram, which changes the argument within the logarithm by a

factor of 10^{12} , changes the time to fall by only $\sim 17\%$. Even if the mass of an electron is used, the time reduces only by about a factor of 3. Nevertheless, Laughlin's formula does have a mass dependence and one may wonder whether this violates Einstein's principle of equivalence. The timescale for motion in a gravitational field would not normally depend upon the inertia. But the source of the mass dependence is purely quantum mechanical. It arises from the irreducible uncertainties in position and/or momentum which apply in the quantum regime.

We derive below the maximum time that the pencil can remain stable using a sequence of approximations in the quantum mechanical treatment. Before turning to this, however, we review the quantum mechanical description of (roughly) localised particles in terms of wave packet solutions of the Schrödinger equation.

2. Wave Packet Solutions Of The Schrödinger Equation In 1D Free Space

In the absence of any potential field (i.e., in free space) the Schrödinger equation in one spatial dimension reduces to,

$$-\frac{\hbar^2}{2m}\partial_x^2\psi = i\hbar\partial_t\psi \quad (2)$$

The plane wave solutions of definite energy, E , and momentum, $\hbar k$, are,

$$\psi \propto e^{i(kx - \omega t)} \quad \text{where, } E = \hbar\omega = \frac{\hbar^2 k^2}{2m} \quad (3)$$

To localise the particle, its momentum must become uncertain. In other words a solution which is a superposition of the definite- k plane waves must be considered. An analytically convenient superposition to consider is a normally distributed momentum, i.e.,

$$\psi_{\text{gwp}} = A \int_{-\infty}^{\infty} e^{-\beta k^2} e^{i(kx - \omega t)} dk \quad (4)$$

where 'gwp' stands for "Gaussian wave packet", $1/2\beta = 2(\Delta k)^2$ is the variance of the momentum distribution, and ω is understood to be given in terms of the wavenumber, k , by (3). Since the exponent is quadratic in k , the integral may be carried out explicitly in the usual manner (e.g. by completing the square and shifting the origin in k -space as appropriate). 'A' is a normalisation constant and may be found by requiring the square-integral over x to be unity. The result is,

$$\psi_{\text{gwp}} = \frac{e^{-\left(\frac{x}{2\alpha}\right)^2}}{(2\pi)^{1/4} \sqrt{\Delta x_t}} \quad (5)$$

where,

$$\alpha = \sqrt{[\beta + i\hbar t/2m]}, \quad \text{and} \quad \Delta x_t = \sqrt{\Delta x_0^2 + (\hbar t/2m\Delta x_0)^2}, \quad \text{and} \quad \beta \equiv \Delta x_0^2 \quad (6)$$

We shall see that Δx_t is a measure of the uncertainty in position at time t , and Δx_0 is the uncertainty in position at time zero. Note that $\Delta x_0 \equiv 1/\Delta k$, i.e. the standard deviation in position is the reciprocal of the standard deviation in momentum. We also note that,

$$|\alpha|^2 = \Delta x_0 \Delta x_t \quad (7)$$

We also note that the minimum uncertainty in momentum is given by,

$$\Delta p_0 = \hbar/2\Delta x_0 \quad (8)$$

In terms of which the positional uncertainty at time t can be written,

$$\Delta x_t = \sqrt{\Delta x_0^2 + (\Delta p_0 t/m)^2} \quad (9)$$

Thus, we see that the initial uncertainty in position is increased by a term which, quite reasonably, multiplies a velocity uncertainty by time – though the two are combined in quadrature rather than linearly, as a consequence of the random-walk nature of the underlying physics.

Note that we may also define a complex positional uncertainty,

$$\Delta x_t^C = \Delta x_0 + i\Delta p_0 t/m = \Delta x_0 + i\hbar t/2m\Delta x_0, \text{ so that } |\Delta x_t^C| \equiv \Delta x_t \quad (9b)$$

Replacing Δx_t in the denominator of (5) by Δx_t^C changes ψ_{gwp} only by a phase factor, and hence is equally valid. In this form we see that (5) is identical to Schiff's Equ.(12.2).

Using,

$$\frac{1}{\alpha^2} + \frac{1}{\alpha^{*2}} \equiv \frac{2}{\Delta x_t^2} \quad (10)$$

the square modulus of (5) gives the probability density as,

$$|\Psi_{\text{gwp}}|^2 = \frac{e^{-x^2/2\Delta x_t^2}}{\sqrt{2\pi} \cdot \Delta x_t} \quad (11)$$

in agreement with Schiff (12.21). Equ.(11) justifies the claim made above that Δx_t is a measure of the uncertainty in position at time t (namely the standard deviation).

Because of the way it has been constructed, the Gaussian wave packet has a probability density which is maximum at $x = 0$ at all times. Equ.(9) shows that the uncertainty in position increases monotonically with time, t . The Gaussian wave packet in x -space becomes progressively flatter and broader with time.

Equ.(8) and (9) display an important pathology. If we attempt to reduce the positional uncertainty by allowing $\Delta x_0 \rightarrow 0$, we are defeated immediately because, for any non-zero time, the positional uncertainty is then infinite. This is because our insistence on $\Delta x_0 \rightarrow 0$ leads to the momentum uncertainty being infinite, so the position becomes completely unknown an instant later. How, then, can we minimise the positional uncertainty at time t ? Simple differentiation shows that this requires the choice $\Delta x_0 = \sqrt{\hbar t / 2m}$, for which choice the spatial and momentum parts of (9) contribute equally, giving the minimum achievable positional uncertainty at time t to be,

$$\Delta x_t^{\min} = \sqrt{\frac{\hbar t}{m}} \quad (12)$$

There is therefore nothing we can do to stop the increasing loss of information about the whereabouts of the particle. The particle inevitably diffuses away from its starting position (near $x = 0$), the word ‘diffuses’ being an accurate description of the physical process involved. The diffusive (random walk) nature of the process is betrayed by the square-root time dependence in (12).

From Tables of the error function, and using (11), there is ~50% chance of the particle being within $|x| < \Delta x_t / \sqrt{2}$ at time t . We may ask the question, “if we draw a circle (sphere) of radius ‘ a ’ around the origin, $x = 0$, what is the shortest length of time that we must wait before the particle has probably diffused outside the sphere?” Of course we are considering just one spatial dimension at present, so the ‘sphere’ is just the interval $[-a, a]$. The most prolonged localisation of the particle pertains for the choice (12), so the greatest time for which the particle remains within the sphere with a probability of >50% is,

$$\frac{\Delta x_t^{\min}}{\sqrt{2}} = \sqrt{\frac{\hbar t_{50\%}}{2m}} = a, \quad \text{hence} \quad t_{50\%} = \frac{2ma^2}{\hbar} \quad (13)$$

Thus, if we imagine the inside of the sphere to contain a vanishingly small potential ‘hill’, Equ.(13) gives the time it takes for the particle to fall off this hill. Thus, the particle ‘falls off’ even though there is nothing to fall from (the potential hill is of zero height).

However, for macroscopic objects, the time given by (13) is extremely long. For example, a one micro-gram particle takes $\sim 2 \times 10^{19}$ seconds (45 times the age of the universe) to diffuse out of a 1mm sphere. On the other hand, an electron would diffuse out of an atom sized sphere in only $\sim 10^{-16}$ seconds. Thus, with no potential to drive the process, the diffusion is insignificant for macroscopic objects. In contrast, even without any potential, the diffusion is crucially important for atom-sized objects. In fact, sub-atomic particles can diffuse over macroscopic dimensions very quickly. Hence a free electron will diffuse ~ 8 mm in one second, and a free proton will diffuse about 0.2mm.

3. The Method of Stationary Phase

Before considering the problem of the balanced pencil we shall first derive the key equation which will be used to approximate the integral which will appear in the problem. This is the method of stationary phase. We shall derive a higher order approximation here than is normally used. This is because this higher order approximation is necessary to get an acceptable result for the free particle wave-packet, whose exact solution has been discussed in Section 2. Ironically, it is a higher degree of approximation than is needed for the balanced pencil problem – at least as we develop it here.

The method of stationary phase is concerned with finding approximations to integrals

of the form $\int_a^b g(x) \cdot \exp\{if(x)\} \cdot dx$, where g and f are real. The implicit assumption is

that g does not vary too much when f changes by $\sim\pi$. In this case, the relatively rapid variation of the phase factor means that contributions from neighbouring regions tend to cancel out in the integral. The exception is when $f(x)$ passes through a stationary value, say at $x = x_0$. In this case there will be constructive, rather than destructive, interference in the vicinity of x_0 . The integral will then approximate to $g(x_0)$ times some ‘width’, $x_2 - x_1$, within which f does not differ too greatly from its stationary value. The approximation is developed by re-writing $g(x)$ as an exponential, i.e.,

$$\mathfrak{I} = \int_a^b \exp\{h(x)\} \cdot \exp\{if(x)\} \cdot dx \quad (14)$$

where h and f are real, and the interval $[a, b]$ is assumed to contain a single stationary point where $\frac{\partial f}{\partial x} = 0$, at $x = x_0$. Expand both functions in a Taylor series to quadratic order,

$$h(x) = h + h'(x - x_0) + h'' \frac{(x - x_0)^2}{2} + \dots \quad (15a)$$

$$f(x) = f + f'' \frac{(x - x_0)^2}{2} + \dots \quad (15b)$$

where, $h = h(x_0)$, $h' = \left. \frac{\partial h}{\partial x} \right|_{x_0}$, $h'' = \left. \frac{\partial^2 h}{\partial x^2} \right|_{x_0}$, $f = f(x_0)$ and $f'' = \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_0}$. Substituting

these expressions into (14) and carrying out the Gaussian integral by completing the square in the exponent, in the usual way, yields,

$$\mathfrak{I} \approx \sqrt{\frac{2\pi}{-h'' - if''}} \cdot \exp\left\{h + if - \frac{h'^2}{2(h'' + if'')}\right\} \times \text{constant phase factor} \quad (16a)$$

providing that $h'' < 0$.

The more usual, lower order, method of stationary phase ignores the Taylor terms in h beyond the constant term, and hence produces,

$$\mathfrak{T} \approx \sqrt{\frac{2\pi}{f''}} \cdot \exp\{h + if\} \times \text{constant phase factor} \quad (16b)$$

To illustrate the method we now look again at the Gaussian wave-packet defined by Equ.(4), which is of the required form, i.e. of the form of Equ.(14) – though the independent variable is k rather than x . We readily see that in this case,

$$\begin{aligned} f(k) &= kx - \frac{\hbar tk^2}{2m}, & k_0 &= \frac{mx}{\hbar t}, & f &= \frac{mx^2}{2\hbar t}, & f'' &= -\frac{\hbar t}{m} \\ h(k) &= -\beta k^2, & h &= -\beta k_0^2 = -\beta \left(\frac{mx}{\hbar t}\right)^2, & h' &= -2\beta \frac{mx}{\hbar t}, & h'' &= -2\beta \end{aligned} \quad (17)$$

The last of these meets the requirement for the approximation of Equ.(16a) to apply. Substituting the values from (17) into (16a) confirms that the approximation reproduces the exact expression given by Eqs.(5,6), up to the normalisation constant.

In passing we note that using the more usual, cruder, approximation given by Equ.(16b), fails to reproduce the exact integral. Instead it produces a result like Equ.(5) but with $\alpha = \Delta x_t = \hbar t / 2m\Delta x_0$, and $\beta \equiv \Delta x_0^2$. This is a reasonable approximation when $t > 2m\Delta x_0^2 / \hbar$, but fails to correctly reflect the fact that the uncertainty in position has a minimum of Δx_0 at $t = 0$.

4. The Balanced Pencil

Let the half-height of the pencil be L , with mass m . Its potential energy when at an angle θ to the vertical is thus $mgL \cos \theta$. Assuming that the bottom end of the pencil does not slip, the kinetic energy due to the centre of mass motion as the pencil falls is $m(L\dot{\theta})^2 / 2$. But there is also kinetic energy of rotation which evaluates to $m(L\dot{\theta})^2 / 3$, making a total kinetic energy of $5m(L\dot{\theta})^2 / 6$. In terms of the linear momentum associated with the centre of mass motion, $p = mL\dot{\theta}$, the total kinetic energy is $5p^2 / 6m$. Taking p to become $\partial / \partial(L\theta)$ when quantised, the Schrödinger equation for our falling pencil is,

$$\left[-\frac{5}{6} \frac{\hbar^2}{L^2 m} \frac{\partial^2}{\partial \theta^2} + mgL \cos \theta \right] \Psi = E\Psi \quad (18a)$$

This differs from the Schrödinger equation for a point mass, m , rolling down a circular hill of radius L only in the factor of $5/6$, which would be $1/2$ for the point mass. We shall work in a quadratic approximation for the potential, giving,

$$\left[-\frac{5}{6} \frac{\hbar^2}{L^2 m} \frac{\partial^2}{\partial \theta^2} + mgL \left(1 - \frac{\theta^2}{2} \right) \right] \Psi = E\Psi \quad (18b)$$

This leads to the potential falling to zero (i.e. the pencil lying flat on the desk) at $\theta = \sqrt{2}$ as opposed to the correct angle of $\theta = \pi/2$. We shall also write (18b) as,

$$\left[-\frac{1}{2} \frac{\hbar^2}{L^2 \tilde{m}} \frac{\partial^2}{\partial \theta^2} + mgL \left(1 - \frac{\theta^2}{2} \right) \right] \Psi = E \Psi \quad (18c)$$

where, $\tilde{m} = \frac{3}{5} m$ for the pencil, but $\tilde{m} = m$ for a point mass rolling down a hill.

The WKB approach holds that the energy eigenstates obeying (18c) can be expressed approximately as,

$$\Psi_k(\theta, t) = A \exp \left\{ i \text{Sign}(k) \int \sqrt{2\tilde{m} \left(E - V_0 \left[1 - \frac{\theta^2}{2} \right] \right)} \cdot \frac{L}{\hbar} d\theta - i \frac{Et}{\hbar} \right\} \quad (19)$$

where the integral appearing in the exponent is an indefinite integral, i.e. θ -dependence is retained, and, $V_0 = mgL$ and $E = \hbar^2 k^2 / 2\tilde{m}$. Thus, k is the wave-number which applies for a particle of energy E in regions where the potential disappears.

Following Equ.(4) we can find a wave-packet solution to (18c) by forming a Gaussian superposition of propagating wave states of type (19), as follows,

$$\Psi_{\text{gwp}} = A \left[\int_{k_M}^{\infty} \exp \left\{ -\beta(k - k_M)^2 \right\} \Psi_k(\theta, t) \cdot dk \right] + \left[\int_{-\infty}^{-k_M} \exp \left\{ -\beta(k + k_M)^2 \right\} \Psi_k(\theta, t) \cdot dk \right] \quad (20)$$

where $k_M = \sqrt{2\tilde{m}V_0} / \hbar$. The integral is carried out over $|k| \geq k_M$ which ensures that all the contributing eigenstates are propagating waves. It is not mathematically obvious that we should exclude states with $|k| < k_M$, which correspond to waves which propagate normally in regions of low potential, but which are exponentially decaying (or rising) near the top of the potential ‘hill’. Nevertheless, we stick with (20) on the practical grounds that the method of stationary phase would not be applicable for real exponents. On physical grounds we claim it is reasonable to ignore waves with $|k| < k_M$ since these are classically forbidden to reside on top of the potential hill. Since we are interested in precisely that, i.e. a classical object at the top of the hill, it seems reasonable to neglect their contribution. We note, though, that there is no guarantee that a Gaussian wave-packet like (20) represents the optimal balance for the pencil. We are essentially just proceeding by analogy with the free wave-packet (for which the Gaussian wave-packet of Equ.(4) is indeed optimal).

The method of stationary phase can now be applied to evaluate (20) approximately. In the notation of Section 3 we have,

$$h(k) = -\beta(k - k_M)^2 \quad (\text{assuming } k > k_M), \quad \text{and}, \quad (21a)$$

$$f(k) = \int \sqrt{\left(\hbar^2 k^2 - \hbar^2 k_M^2 \left[1 - \frac{\theta^2}{2} \right] \right)} \cdot \frac{L}{\hbar} d\theta - \frac{\hbar t}{2\tilde{m}} k^2 \quad (21b)$$

The integral in (21b) can be evaluated exactly, and ignoring the constant term gives,

$$f(k) = \frac{\theta L}{2\hbar} [a^2 + b^2 \theta^2]^{1/2} + \frac{a^2 L}{2b\hbar} \log \left\{ \xi + \sqrt{1 + \xi^2} \right\} - \frac{\hbar t}{2\tilde{m}} k^2 \quad (22a)$$

$$\text{where,} \quad a^2 = \hbar^2 (k^2 - k_M^2), \quad b^2 = \frac{\hbar^2 k_M^2}{2}, \quad \xi = \frac{b\theta}{a} \quad (22b)$$

The method of stationary phase now requires us to find the stationary point of $f(k)$ with respect to k . To simplify this we note that the log term will only be very weakly dependent upon k , and hence we shall treat the log term as a constant. To do so we put $\theta \sim 1$ inside the log. Specifically we define,

$$a_0^2 = \hbar^2 (k_0^2 - k_M^2), \quad \xi_0 = \frac{b}{a_0}, \quad \aleph = \log \left\{ \xi_0 + \sqrt{1 + \xi_0^2} \right\} \quad (22c)$$

where k_0 is the value of k which makes f stationary in this approximation (see below), so that \aleph is just a numerical constant. We thus approximate (22a) by,

$$f(k) = \frac{\theta L}{2\hbar} [a^2 + b^2 \theta^2]^{1/2} + \aleph \frac{La^2}{2\hbar b} - \frac{\hbar t}{2\tilde{m}} k^2 \quad (22d)$$

If this approximation is thought rather too coarse, we shall show later that, for the mass and length scales of interest in a macroscopic application (specifically a pencil), a factor of 10 error in the argument of the logarithm represents an error in the stability time of less than 2%. Differentiating (22d) with respect to k gives,

$$\frac{\partial f}{\partial k} = \hbar k \left\{ \frac{L\theta}{2\sqrt{a^2 + b^2 \theta^2}} + \frac{\aleph L}{b} - \frac{t}{\tilde{m}} \right\} \quad (23)$$

Hence, f is stationary when $k = k_0$ where k_0 is found from,

$$\frac{\hbar x}{2\sqrt{a_0^2 + b^2 x^2}} = \frac{t}{\tilde{m}} - \frac{\aleph \hbar}{b} \quad (24)$$

$$\text{Solving gives,} \quad k_0^2 = k_M^2 \left(1 - \frac{\theta^2}{2} \right) + \left(\frac{L\theta}{2} \right)^2 \left(\frac{\hbar t}{\tilde{m}} - \frac{\sqrt{2\aleph L}}{k_M} \right)^{-2} \quad (25)$$

It suffices to work in lowest order of the method of stationary phase, using Equ.(16b) for which the real exponent part is $\propto \exp\{h\}$. Using (21a) this gives,

$$\Psi_{\text{gwp}} \propto \exp\left\{-\beta(k_0 - k_M)^2\right\} \quad (26)$$

$$\text{Write (25) as, } k_0^2 = k_M^2 + \frac{\theta^2}{\Delta_\theta^2}, \text{ where, } \Delta_\theta^{-2} = \left(\frac{L}{2}\right)^2 \left(\frac{\hbar t}{\tilde{m}} - \frac{\sqrt{2}\aleph L}{k_M}\right)^{-2} - \frac{k_M^2}{2} \quad (27)$$

In (27) we cannot consider times earlier than $t_{\min} = \frac{\sqrt{2}\aleph L \tilde{m}}{\hbar k_M}$ because otherwise there is no solution for k_0 from Equ.(24) and the method of stationary phase does not apply.

At times just infinitesimally greater than t_{\min} we see from (27) that the first term blows up and we have $\Delta_\theta = 0$. Thus, the wave-packet solution does correspond to a highly localised particle at early times, as required. (The fact that $\Delta_\theta = 0$ rather than a small but finite value may be traced to the approximations used, particularly the first order method of stationary phase, which also leads to vanishing positional uncertainty at early times in the case of a free particle, see Section 3).

However, from (27) we see that there is a finite time when the RHS becomes zero, and hence $\Delta_\theta \rightarrow \infty$. This is when,

$$\left(\frac{L}{2}\right) \left(\frac{\hbar t}{\tilde{m}} - \frac{\sqrt{2}\aleph L}{k_M}\right)^{-1} = \frac{k_M}{\sqrt{2}} \quad (28a)$$

Solving gives,

$$t_{\text{fall}} \leq \left(\frac{1}{\sqrt{2}} + \sqrt{2} \cdot \aleph\right) \frac{\tilde{m}L}{\hbar k_M} \quad (28b)$$

The notation reflects the fact that when the uncertainty in position diverges the pencil can be regarded as having already fallen over. Hence the time given by (28a,b) is the upper bound stability time. Substituting for k_M gives,

$$t_{\text{fall}} \leq \left(\frac{1}{\sqrt{2}} + \sqrt{2} \cdot \aleph\right) \sqrt{\frac{\tilde{m}}{2V_0}} \cdot L = \left(\frac{1}{\sqrt{2}} + \sqrt{2} \cdot \aleph\right) \sqrt{\frac{\tilde{m}}{m} \cdot \frac{L}{2g}} \quad (28c)$$

In this derivation we have rather obscured the nature of the positional dependence of the wavefunction. The exponent in Equ.(26) actually depends upon $(k_0 - k_M)^2$, not upon k_0^2 as we have implicitly assumed. The true exponent can be written, using (27),

$$(k_0 - k_M)^2 = 2k_M^2 + \frac{\theta^2}{\Delta_\theta^2} - 2k_M \sqrt{k_M^2 + \frac{\theta^2}{\Delta_\theta^2}} \quad (29a)$$

At sufficiently early times, when Δ_θ is small, the term $\frac{\theta^2}{\Delta_\theta^2}$ dominates, and the above derivation is valid. However, we are most interested in the time when the pencil becomes unstable, at which time Δ_θ is very large and the k_M terms dominate. By retaining three terms in the binomial expansion of the square-root, Equ.(29a) becomes in this limit,

$$(k_0 - k_M)^2 \approx \frac{1}{4k_M^2} \left(\frac{\theta}{\Delta_\theta} \right)^4 \quad (29b)$$

Thus, the spatial dependence of the wave function as instability is approached tends to the form $\exp\left\{-\frac{\theta^4}{\Delta_{\theta t}^4}\right\}$, where, $\Delta_{\theta t} = \frac{\sqrt{2k_M}}{\beta^{1/4}} \Delta_\theta$. Thus, the spatial dependence is rather unusual and the ‘characteristic scale’, $\Delta_{\theta t}$, differs from the scale Δ_θ by a constant factor. However, this makes no significant difference to the argument. The spatial uncertainty still blows up when Δ_θ diverges, at the time given by Equ.(28c). Thus Equ.(28c) is indeed the valid upper bound stability time.

The final step is to approximate ξ_0 so that \aleph can be estimated from (22c). For this we note that (27) gives,

$$a_0^2 = \hbar^2 (k_0^2 - k_M^2) = \frac{\hbar^2 \theta^2}{\Delta_\theta^2} \quad (30a)$$

and hence,
$$\xi_0 \approx \frac{b}{a_0} = \frac{\hbar k_M}{\sqrt{2}} \cdot \frac{\Delta_\theta}{\hbar \theta} \approx k_M L \quad (30b)$$

where we are ignoring factors of order unity and have replaced the spatial uncertainty with a value representative of near-instability ($\Delta_\theta \sim L$), and correspondingly put $\theta \sim 1/\sqrt{2}$. Substituting the value of k_M gives,

$$\xi_0 \sim \sqrt{\frac{\tilde{m}}{m}} \cdot m \sqrt{2g} \cdot \frac{L^{3/2}}{\hbar} \quad (30c)$$

ξ_0 is dimensionless, and for macroscopic objects is an extremely large number because of the denominator of \hbar . Consequently, for macroscopic objects we can make the further approximations,

$$\aleph = \log\left\{\xi_0 + \sqrt{1 + \xi_0^2}\right\} \approx \log(2\xi_0) = \frac{1}{2} \log(4\xi_0^2) \quad (31a)$$

$$\frac{1}{\sqrt{2}} + \sqrt{2} \cdot \aleph \approx \sqrt{2} \cdot \aleph \quad (31b)$$

Hence, our final expression for the stability time is,

$$t_{\text{fall}} \leq \sqrt{\frac{\tilde{m}}{m}} \cdot \sqrt{\frac{L}{4g}} \cdot \log \left\{ \frac{\tilde{m}}{m} \cdot \frac{8gm^2L^3}{\hbar^2} \right\} \quad (32)$$

In the case of a ball rolling down a hill of height L , for which $\tilde{m} = m$, this is of the same form as Laughlin's expression, Equ.(1). In particular the argument of the logarithm is the same. However, our expression has a factor of 4 in the denominator of the leading term, rather than 32. Equ.(1) therefore appears to be a factor of $\sqrt[4]{8}$ too small (unless I have made a blunder). Substituting Laughlin's values ($m = 7.3\text{kg}$, $L=1\text{m}$), Equ.(32) gives an upper bound stability time for a bowling ball atop a 1 metre circular hill of 26 seconds.

The Pencil: A typical pencil has mass 5g (0.005 kg) and half-length, $L = 8\text{cm} = 0.08\text{m}$. Substitution into Equ.(32) gives an upper limit stability time of **~5 seconds**. We have included the terms in $\tilde{m}/m = 0.6$ in this case.

The approximations we have made are so crude that the numerical factor of 8 within the logarithm in Equ.(32) is actually meaningless. It could easily be in error by a factor of 4 or more. The agreement of argument of the logarithm with Laughlin's formulae, Equ.(1), is therefore probably fortuitous. However, for macroscopic objects an error in this argument by a factor of ten or so makes very little difference to the answer. The argument of the logarithm for the pencil is 5.42×10^{61} . Increasing or decreasing this by a factor of ten causes the stability time to change by less than 2%.

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