Thermal Stresses In A Thin Vessel With A Small Hole
(Axisymmetric Temperature Change Does Not Change Hole Size)

Two loadings are considered:

1) Heat transfer on the bore of the vessel, but none on the bore of the hole. The temperature distribution is assumed to produce a pure bending stress through the wall of the vessel (i.e. for a thin shell, a linear temperature distribution).

2) Heat transfer on the bore of the hole, but none on the bore of the vessel.

<u>Approximation</u>: The vessel is treated as an infinite flat plate. This is why it has to be a *small* hole and a *thin* vessel. Both loadings are then axisymmetric about the centre of the hole. The plate is restrained from bending in order to produce the thermal stress. This is equivalent to a uniform bending moment per unit length being applied around the whole perimeter of the plate. The remote stresses are equi-biaxial bending.

The in-plane coordinates are (x, y) or (r, θ), with r = 0 at the centre of the hole. The through-thickness coordinate is z.

<u>Loading 1:</u>

The equilibrium equations in cylindrical coordinates are,

$$\sigma_{r,r} + \frac{1}{r}\tau_{r\theta,\theta} + \tau_{rz,z} + \frac{\sigma_r - \sigma_\theta}{r} = 0; \qquad \frac{1}{r}\sigma_{\theta,\theta} + \tau_{r\theta,r} + \tau_{\theta z,z} + \frac{2}{r}\tau_{r\theta} = 0 \qquad (1, 2)$$

$$\sigma_{z,z} + \frac{1}{r}\tau_{\theta z,\theta} + \tau_{rz,r} + \frac{1}{r}\tau_{zr} = 0 \qquad (3)$$

(the comma denotes a derivative). My contention is that the plate can be considered as comprised of a very large number of very thin laminae and that each lamina is stressed independently of the rest in equi-biaxial membrane loading. Hence the SCF is exactly 2.

<u>Proof</u>: By axi-symmetry, the θz and θr shear stresses are zero. The faces of the plate are unconstrained and unloaded, so the out-of-plane (z) stress is zero. [Equivalently, if the problem is plane strain, then the term $\sigma_{z,z}$ is still zero]. The equilibrium equations thus become,

$$\sigma_{r,r} + \tau_{rz,z} + \frac{\sigma_r - \sigma_\theta}{r} = 0; \qquad \frac{1}{r}\sigma_{\theta,\theta} = 0; \qquad \tau_{rz,r} + \frac{1}{r}\tau_{zr} = 0 \qquad (1b, 2b, 3b)$$

Thus, only the “inter-lamina shear” τ_{rz} remains as the potential ‘glue’ which could couple the behaviour of the laminae together. However, the last equation has the general solution,

$$\tau_{rz} = \frac{C}{r} \qquad (4)$$

for some constant 'C'. However, this shear stress must be zero at the free bore of the hole, at r = a. Hence we must have C = 0, and thus the inter-lamina shear is zero everywhere. The equilibrium equations are thus uncoupled, becoming simply,

$$\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1c)$$

Noting that Equ.(1b) just states that the hoop stress is axisymmetric.

Note that we do not need to consider displacement compatibility because the formulation of the problem ensures that the total displacement is everywhere zero, i.e. the elastic strain is everywhere equal and opposite to the applied thermal strain. The linear variation of the thermal strain through the thickness thus implies that the elastic strain, and hence stress, varies linearly through-thickness

The solution can thus be written down immediately[1],

$$\sigma_r = \sigma_0\left[1 - \frac{a^2}{r^2}\right]; \qquad \sigma_\theta = \sigma_0\left[1 + \frac{a^2}{r^2}\right]; \qquad \text{where, } \sigma_0 = \frac{2z}{t}\sigma_b \qquad (5)$$

where t is the thickness, and z is measured from the section centre, and σ_b is the outer fibre bending stress remote from the hole. Note that the radial stress is zero at the hole (r = a), and that the stress becomes equi-biaxial ($\sigma_r = \sigma_\theta = \sigma_0$) remote from the hole.

Most importantly, the hoop stress concentration factor at the hole is exactly 2.

Loading (2):

The concept of SCF is not relevant in this case because the remote stress is zero. The equations of equilibrium, compatibility and Hooke's law reduce (for plane stress) to,

$$\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} = 0; \qquad Eu_{,r} = \sigma_r - \nu\sigma_\theta + E\alpha T; \qquad E\frac{u_r}{r} = \sigma_\theta - \nu\sigma_r + E\alpha T \qquad (6)$$

where u is the radial displacement and T = T(r) is the temperature change at radius r.

Provided that the plate is infinite and that the temperature change, T(r), falls to zero at large r, it can be shown that the size of the hole (r = a) is unchanged by the change in temperature. This means that the surface of the hole is fully constrained (the elastic strain is equal and opposite to the thermal strain) and hence that the hoop stress on the bore of the hole is $-E\alpha T(a)$. Oddly, whilst this result for the stress is what one would expect, the fact that the hole does not change size – which is actually an equivalent statement – is not intuitively obvious (to me, anyway).

Note that the stress at the hole depends only upon the temperature change at the hole – not on the rest of the temperature distribution or on the temperature gradient. This only applies for an infinite plate with $T(r \to \infty) \to 0$.

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[1] for which read 'after two further glasses of wine'

<u>Proof</u>: Multiplying the last of Equs.6 by r and differentiating gives,

$$Eu_{r,r} = \sigma_\theta - \nu\sigma_r + E\alpha T + r\partial_r\left[\sigma_\theta - \nu\sigma_r + E\alpha T\right] \qquad (7)$$

Equating this to the second of Equs.6 and re-arranging gives,

$$(1+\nu)(\sigma_r - \sigma_\theta) = r\partial_r\left[\sigma_\theta - \nu\sigma_r + E\alpha T\right] = -(1+\nu)r\sigma_{r,r} \qquad (8)$$

where the last step follows from the first of Equs.6. Dividing out r and integrating gives,

$$-(1+\nu)\sigma_r = \sigma_\theta - \nu\sigma_r + E\alpha T + \text{constant} \qquad (9)$$

i.e.
$$\sigma_r + \sigma_\theta = -E\alpha T + \text{constant} \qquad (9b)$$

Now provided that the plate is infinite and the temperature change falls to zero at large distances, the stresses and T are all zero at large r and hence the constant term is zero. Hence,

$$\sigma_r + \sigma_\theta = -E\alpha T \qquad (9c)$$

This gives the hoop stress on the bore of the hole as $\sigma_\theta = -E\alpha T$, as claimed, since the radial stress is zero there. Substituting (9c) into the last of Equs.6 gives,

$$E\frac{u}{r} = -(1+\nu)\sigma_r \qquad (10)$$

Hence, the fact that the radial stress is zero on the bore of the hole implies that u = 0 on the bore of the hole, i.e. the hole does not change size. QED.

Note that Equ.9c states that the hydrostatic stress is everywhere equal to -EαT. To find the individual stress components is harder work, requiring Equ.9c to be substituted into the first of Equs.6 and solving the differential equation, giving,

$$\sigma_r = \frac{1}{r^2}\int_a^r E\alpha T.rdr \qquad (11)$$

and then the hoop stress follows from Equ.9c.

All the above results continue to hold if the coefficient of thermal expansion is an arbitrary function of radius (but axisymmetric). This is because α occurs only in the combination αT. Thus, if the plate consists of annular rings of different materials all the results still hold *provided that* radial gaps do not open between adjacent rings.