Feynman's Disc Paradox

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In Volume II, chapter 17, §17-4 of the famous Feynman lectures, he introduces a paradox. The situation is depicted below,

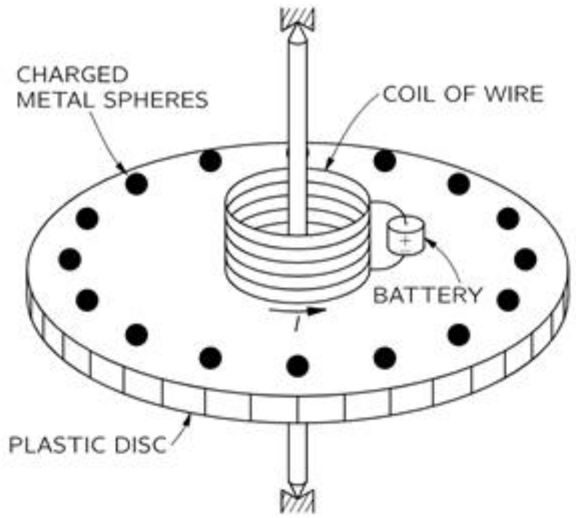

Fig. 17–5. Will the disc rotate if the current I is stopped?

Initially the disc is stationary. It has charges fixed around its periphery, which are also stationary. There is a coil, or solenoid, whose centre-line lies on the axle through the disc. The disc is free to turn about bearings which support the axle above and below. Initially there is a steady current in the coil, thus producing a steady magnetic field – approximating a magnetic dipole field. There is no force acting on the charges initially.

At time zero the source of electrical power driving the current is disconnected. The current, and hence the magnetic field, drop quicky to zero. A changing magnetic field produces an induced electric field through the Maxwell equation $\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t}$ where $\widetilde{\boldsymbol{E}}$ is the induced electric field (i.e., Faraday's law of induction). Bold type indicates vectors. Axisymmetry (at least in the limit of a circumferentially uniform distribution of charges) and the fact that the magnetic field is vertical at the charges, means that the vertical component of $\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}}$ is non-zero whilst the magnetic field is falling, and has magnitude $\frac{1}{r}\partial_r\left(r\tilde{E}_\theta\right)$. Thus, there is a tangential component to the induced electric field, and hence a tangential force acting on the charges which is the same size and direction all around, and hence produces a torque which will cause the disc to rotate (assuming zero friction).

The paradox is that the situation is initially static, and hence appears to have zero angular momentum, and yet after turning off the current to the coil we gain angular momentum, thus apparently violating the conservation of angular momentum.

The resolution of the paradox lies in the realisation that a static electric field and a static magnetic field possess field momentum, if they are not parallel, according to the Poynting vector $\boldsymbol{N} = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ (the field momentum per unit volume). Here, $\boldsymbol{E}$ is the electric field due to the charges. On the disc this field is radial, whereas the magnetic field is vertical, and so the field momentum is tangential. Hence, the system does possess angular momentum initially even while it is mechanically stationary.

What remains is to confirm that the field angular momentum before time zero equals the mechanical angular momentum after time zero.

The induced field obeys $\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t}$ but the static field of the charges obeys $\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$. Conversely, because the induced field corresponds to no charge and the static field corresponds to no magnetic field, we also have $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ and $\varepsilon_0 \boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{E}} = 0$.

The force acting on the charges, per unit volume, is $\rho \widetilde{\boldsymbol{E}}$ and so the torque is $T = \int \boldsymbol{r} \times \rho \widetilde{\boldsymbol{E}}\, dV$. We can replace the charge density from $\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$ and since torque is rate of change of angular momentum the induced mechanical angular momentum is,

$$\boldsymbol{L}_m = \int \varepsilon_0 (\boldsymbol{\nabla} \cdot \boldsymbol{E}) \boldsymbol{r} \times \widetilde{\boldsymbol{E}}\, dV dt \tag{1}$$

The initial field angular momentum is $\boldsymbol{L}_f = \int \boldsymbol{r} \times (\varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}) dV$. The magnetic field can be re-expressed as a time-integral over the induced electric field from $\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t}$ which gives us,

$$\boldsymbol{L}_f = -\int \boldsymbol{r} \times \left(\varepsilon_0 \boldsymbol{E} \times \left(\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}} \right) \right) dV dt \tag{2}$$

The challenge is to prove that (1) and (2) are equal. This is rather harder work than it has any right to be. Here I follow the approach of Lombardi, Ref.[1].

To conclude that (1) and (2) are equal requires the additional conditions $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{E}} = 0$. These can be incorporated into (1) and (2) by symmetrising both, so,

$$\boldsymbol{L}_m = \varepsilon_0 \int \left[(\boldsymbol{\nabla} \cdot \boldsymbol{E}) \boldsymbol{r} \times \widetilde{\boldsymbol{E}} + \left(\boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{E}} \right) \boldsymbol{r} \times \boldsymbol{E} \right] dV dt \tag{1b}$$

$$\boldsymbol{L}_f = -\varepsilon_0 \int \boldsymbol{r} \times \left[\boldsymbol{E} \times \left(\boldsymbol{\nabla} \times \widetilde{\boldsymbol{E}} \right) + \left(\widetilde{\boldsymbol{E}} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) \right) \right] dV dt \tag{2b}$$

The i component of the integrand of (1b) is $\in_{ijk} x_j \left(E_{J,J} \tilde{E}_k + \tilde{E}_{J,J} E_k \right)$ whereas that of (2b) is $\in_{ijk} x_j \left[E_J \left(\tilde{E}_{J,k} - \tilde{E}_{k,J} \right) + \tilde{E}_J \left(E_{J,k} - E_{k,J} \right) \right]$. Adding the two gives the integrand whose integral we need to prove is zero. Terms simplify as,

$$\in_{ijk} x_j \left[-\left(E_J \tilde{E}_J \right)_{,k} + \left(E_J \tilde{E}_k \right)_{,J} + \left(E_k \tilde{E}_J \right)_{,J} \right] \tag{3}$$

That the volume integral of (3) is zero follows in two steps. The first follows from,

$$\int x \frac{\partial f}{\partial y} dV = 0 \tag{4}$$

Where the integral is over all space and the function f is assumed to fall off at infinity at least as fast as $1/r^3$. This applies here because both $\boldsymbol{E}$ and $\widetilde{\boldsymbol{E}}$ fall off as $1/r^2$ and so their product falls of as $1/r^4$. A heuristic demonstration of (4) is to carry out the y integral at fixed x, z giving $x\big(f(x, \infty, z) - f(x, -\infty, z)\big)$. Providing $f(x, \pm\infty, z)$ reduces at least as fast as $1/x^3$ then the x integral goes as $1/x$ or smaller, and so also gives zero over the infinite domain.

Considering the x component, the only terms now remaining to consider in (3) are,

$$\in_{xyz} y\left[+\left(E_y\tilde{E}_z\right)_{,y} + \left(E_z\tilde{E}_y\right)_{,y}\right] + \in_{xzy} z\left[\left(E_z\tilde{E}_y\right)_{,z} + \left(E_y\tilde{E}_z\right)_{,z}\right]$$

Which can be re-arranged as,

$$(y\partial_y - z\partial_z)\left[E_y\tilde{E}_z + E_z\tilde{E}_y\right]$$

Integration by parts confirms for any function which falls off sufficiently fast, as the above does, then,

$$\int y\partial_y f\, dV = \int z\partial_z f\, dV \qquad (5)$$

over infinite limits. This follows because under this assumption, integration by parts gives,

$$\left(\int y\partial_y f\, dy\right) dxdz = -\int f dydxdz$$

Which therefore equals $\int z\partial_z f\, dV$ which is just the same. **QED**.

Hence the final mechanical angular momentum does indeed equal the initial field angular momentum – phew! The conservation law survives again!

When I first heard of this explanation of the paradox I had a problem with it. I reasoned that the coil could be a long solenoid, in which case the magnetic field would be zero (or as small as we might like) outside it. But if the solenoid, being comprised of conducting wires, effectively shielded its inside from the external charges, there would be no static E-field inside it. Consequently, I argued, the Poynting vector $\boldsymbol{N} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B}$ would be zero everywhere and the initial field angular momentum would be zero. On the other hand, the induced emf at the charges depends only upon the rate of change of total flux linkage within the circle of the charges – and so there would still be an induced torque causing the disc to turn even though the B-field outside the solenoid was zero. So we would get non-zero angular momentum from an initial zero angular momentum – oh dear!

The fallacy of my reasoning, I now realise, is that although the Poynting vector could be as small as we want both inside the solenoid and also outside it up to the circle of charges, what we are interested in is $\boldsymbol{L}_f = \int \boldsymbol{r}\times(\varepsilon_0\boldsymbol{E}\times\boldsymbol{B})dV$ where the integral is over the whole of space. Now lines of B-flux are closed loops, so the vertical upward flux within the solenoid must be matched by an equal vertical downwards flux outside the solenoid somewhere – though that may mean a very small flux density (i.e., B) but extending over very large distances. In other words, the initial field angular momentum may be located a long way from the disc and the charges. One needs to have more respect for infinite integrals than my qualitative argument portrayed. One lives and learns!

References

[1] Gabriel Lombardi. (1983). Feynman's disk paradox. American Journal of Physics 51(3):213-214, March 1983. http://doi.org/10.1119/1.13272 (Available at https://www.researchgate.net/publication/243489921_Feynman's_disk_paradox)