**The Feynman Path Integral: The Closer You Look, The Weirder It Gets**

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We are used to the Feynman path integral giving rise to quantum mechanical behaviour as a consequence of the contributions of all possible paths, including highly erratic, non-differentiable paths. But surely the classical limit reproduces a nice, sensible, smooth, continuous, everywhere differentiable, classical trajectory? No, sorry, it does not, quite the opposite. Such smooth curves can be shown to make no contribution. Their Wiener measure is zero. Instead, the classical limit of the Feynman integral entails non-zero contributions only from paths which are so violently kinked that they are nowhere differentiable. This can be established rigorously [see for example M. Reed and B. Simon: *Methods of modern mathematical physics, vol.2: Fourier analysis, self-adjointness*, Academic Press, San Diego (1975)]. Here we demonstrate this fact by a very simple argument.

The propagator (or Green's function, or two-point function, or partition function, according to taste or subtle dialect) is, for a single particle moving in one dimension,

$$G(x_1, t_1; x_N, t_N) = \int_\Omega \exp\left\{\frac{i}{\hbar} S(x(t))\right\} \cdot Dx(t) \qquad (1)$$

where S(x(t)) is the classical Action for the path x(t) which is arbitrary except that it is constrained to start at $x_1$ and end at $x_N$, i.e., $x(t_1) = x_1$ and $x(t_N) = x_N$. Hence,

$$S(x(t)) = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t) \cdot dt \qquad (2)$$

where L is the classical Lagrangian (i.e. the kinetic energy minus the potential energy, as a function of position and velocity). The integration in (1) is a functional integration over "all paths". In practice this is to be understood as the limit of a large number of ordinary integrals over the position of the particle at intermediate times. Thus, the integration measure is,

$$Dx(t) \equiv dx_2 dx_3 dx_4 ... dx_{N-1} \qquad (3)$$

where $x_i \equiv x(t_i)$, and the intermediate times obey $t_1 < t_2 < t_3 \ldots < t_N$, and may be taken as equally spaced. The range of the integral in (1), Ω, is whatever spatial domain we wish to confine the particle to – and may be infinite. The quantum wavefunction at time $t_N$ is found from that at time $t_1$ from,

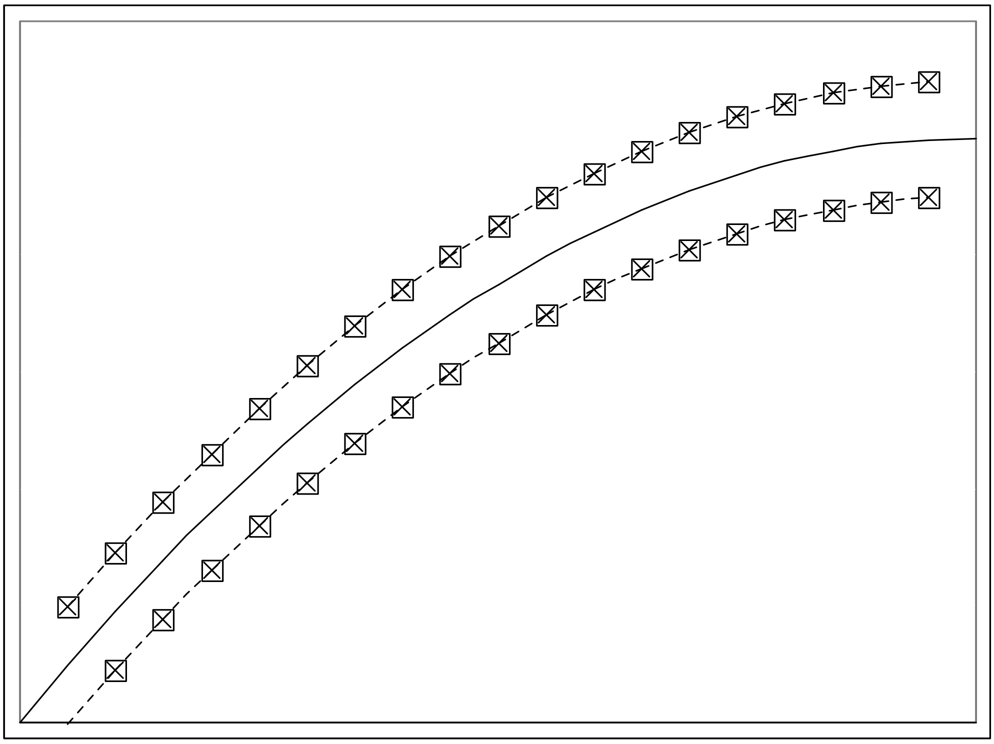
$$\psi(x_N, t_N) = \int_\Omega S(x_N, t_N; x_1, t_1)\psi(x_1, t_1)dx_1 \qquad (4)$$

where (4) is an ordinary integral.

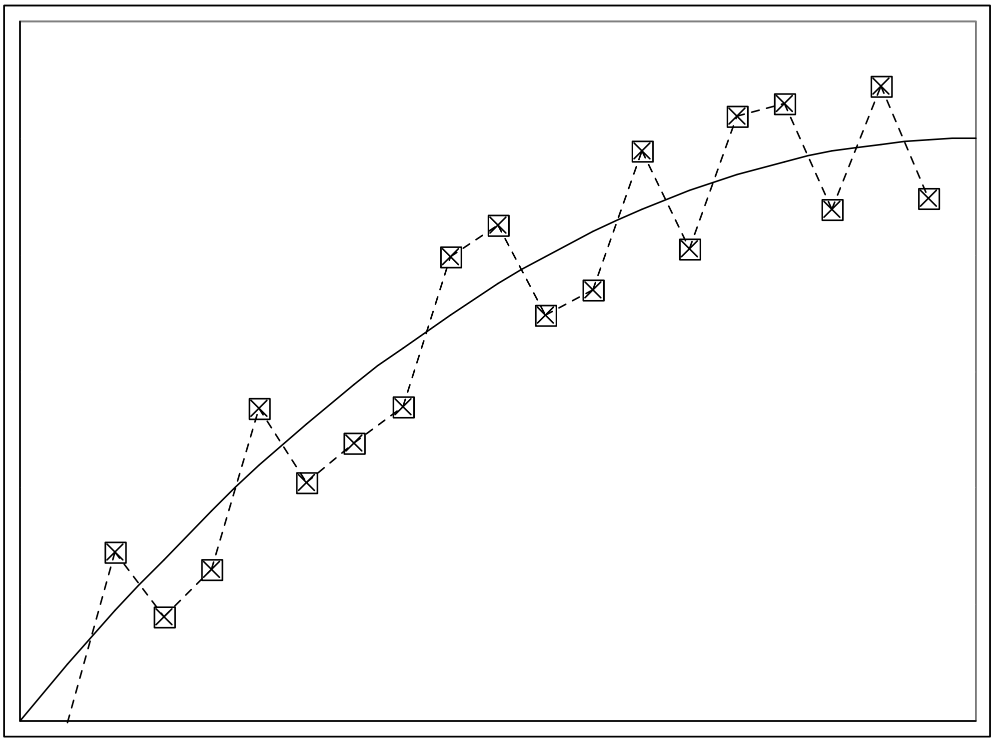
All this machinery is not really needed for our present purpose. The important thing is that all paths, including very erratic, violently kinked, paths contribute to (1) on an equitable basis. The classical limit is argued as follows:-

Suppose we are dealing with a large (i.e. classical) amount of Action. That is, $S >> \hbar$. If that is the case then, in general, even a small change in the path will induce a change in S which is larger than $\hbar$. Hence, the phase $S/\hbar$ varies wildly by amounts which are large compared with unity, even for very small changes to the path. Hence neighbouring paths will tend to cancel out in the path integral, (1). The exception to this is when the path in question is the classical trajectory. By definition, this is the path which gives a stationary Action, i.e., $\delta S/\delta x(t) = 0$. Hence, small path changes about the classical trajectory produce little phase change, and instead of cancelling out they add constructively. This is Feynman's argument which shows that, for large amounts of Action, the only paths which contribute significantly are those close to the trajectory which makes the Action stationary, i.e. the classical path.

However, it does not follow from this that the paths which contribute are smooth and differentiable. This can be seen simply by considering that subset of paths which are at the same distance $\varepsilon$ from the classical trajectory at all the intermediate times $t_2$, $t_3$, …$t_{N-1}$. The paths differ because at each of these times the particle may be at either $x + \varepsilon$ or $x - \varepsilon$. The two paths which are nice and smooth are the ones which always lie either above or below the classical trajectory:-



However, these are obvious very special paths. Out of the $2^{N-2}$ paths which can be composed out of the N-2 time points, ($2^{N-2} - 2$) of them involve crossing over the classical trajectory. A typical path chosen at random is,



For large enough N, virtually all paths are of this kinked type. As N becomes very large, the distance between kinks vanishes. Consequently, the only paths which make a non-zero contribution to the Feynman path integral are those which are non-differentiable everywhere. They consist of "nothing but corners".

Strictly we have not established this rigorously because we have not demonstrated that the weighting provided by the factor $\exp\left\{\frac{i}{\hbar}S\right\}$ does not so favour the smooth paths over the kinked paths that the numerical advantage of the kinked paths is undermined. However, this can be done.

Hence, even in the classical limit, the path of a particle is a fractal[1], although the amplitude of this fractal about the classical trajectory is very small and hence unobserved.

[1] Actually, it is a superposition of such fractals, of course. So, there is no classical limit really.