

## Euler's Summing of $\sum_{r=1}^{\infty} \frac{1}{r^2}$

Consider the function  $\sin(x)$ . What's that got to do with it? You may well ask. Pretend  $\sin(x)$  is a finite polynomial. We know that its roots are  $0, \pm\pi, \pm 2\pi, \pm 3\pi \dots$  etc. So we can write it as,

$$\sin x = \text{constant} \times \dots (x + 3\pi)(x + 2\pi)(x + \pi)x(x - \pi)(x - 2\pi)(x - 3\pi)\dots \quad (1a)$$

But we know that  $\frac{\sin x}{x} \rightarrow 1$  as  $x \rightarrow 0$ , so the constant must be the reciprocal of  $\dots(3\pi)(2\pi)(\pi)(-\pi)(-2\pi)(-3\pi)\dots$ , and so we get,

$$\begin{aligned} \sin x &= \dots \left(1 + \frac{x}{3\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 + \frac{x}{\pi}\right) x \left(1 - \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 - \frac{x}{3\pi}\right) \dots \\ &= x \left(1 - \left(\frac{x}{\pi}\right)^2\right) \left(1 - \left(\frac{x}{2\pi}\right)^2\right) \left(1 - \left(\frac{x}{3\pi}\right)^2\right) \dots \end{aligned} \quad (1b)$$

But we know that  $\sin x = x - \frac{x^3}{6} + O(x^5)$ . Equating the coefficient of  $x^3$  in these two expressions gives,

$$-\frac{1}{6} = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \dots = -\frac{1}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \quad (2a)$$

Hence,

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} \quad (2c)$$

But this is surely not rigorous, you cry! Who cares? It's brilliant.

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