Euler's Summing of $\sum_{r=1}^{\infty} \frac{1}{r^2}$

Consider the function sin(x). What's that got to do with it? You may well ask. Pretend sin(x) is a finite polynomial. We know that its roots are 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$ …etc. So we can write it as,

$$\sin x = \text{constant} \times .....(x+3\pi)(x+2\pi)(x+\pi)x(x-\pi)(x-2\pi)(x-3\pi)..... \qquad (1a)$$

But we know that $\frac{\sin x}{x} \to 1$ as $x \to 0$, so the constant must be the reciprocal of $......(3\pi)(2\pi)(\pi)(-\pi)(-2\pi)(-3\pi).....$, and so we get,

$$\begin{aligned}\sin x &= .....\left(1+\frac{x}{3\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1+\frac{x}{\pi}\right)x\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right).....\\ &= x\left(1-\left(\frac{x}{\pi}\right)^2\right)\left(1-\left(\frac{x}{2\pi}\right)^2\right)\left(1-\left(\frac{x}{3\pi}\right)^2\right).....\end{aligned} \qquad (1b)$$

But we know that $\sin x = x - \frac{x^3}{6} + O(x^5)$. Equating the coefficient of x^3 in these two expressions gives,

$$-\frac{1}{6} = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - ... = -\frac{1}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2} \qquad (2a)$$

Hence,

$$\sum_{r=1}^{\infty}\frac{1}{r^2} = \frac{\pi^2}{6} \qquad (2c)$$

But this is surely not rigorous, you cry! Who cares? It's brilliant.