Energy Release Rate: Relationships Between G, J and the Ks (and what do J_2 and J_3 mean, anyway?)

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1. Objectives

A preceding Note has derived the expressions for the vector J integrals, and shown that they are path independent for non-linear elasticity (deformation theory). For the LEFM case the J integrals were evaluated in terms of K_I, K_{II} and K_{III} by direct substitution of the LEFM crack tip fields into the J-integral. In this Note a more elegant, and more physically enlightening, derivation of the expression $EJ_1 = \kappa\left(K_I^2 + K_{II}^2\right) + (1+\nu)K_{III}^2$ is presented. This proceeds by firstly showing that J_1 equals the energy release rate in the general non-linear elastic case (and hence also for proportional, monotonic, loading in plasticity). This follows an argument due to Eshelby (1956). In the LEFM case it is then simple to show that the energy release rate equals the above expression in terms of the Ks. A by-product of Eshelby's argument is that it sheds light on the physical meaning of J_2 and J_3, and explains why they are little used in fracture mechanics. Finally, it is shown that considering the dynamic case reinforces the interpretation of J as the energy release rate, and hence the force acting on the defect.

2. Definition of the Energy Release Rate

Energy release rate, G, is defined as the energy required to create a unit increment of crack area (so it is actually the derivative of energy with respect to crack area, not really a 'rate'). If no energy is supplied by any external agency to the body whilst the crack is extending, and in the quasi-static case, it follows that the energy requirement for the crack extension is met by decreasing the body's strain energy, U. In this case the energy release rate is thus $G = -\frac{\partial U}{\partial A}$. Note that A is the area of one crack face only, not both.

Alternatively, if there are applied loads (as opposed from applied displacements) acting on the body, then the external agency does work when the crack extends (because the increased compliance of the structure leads to movement at the point of application of the loads). Moreover, if the applied loads dominate over the applied displacements, the strain energy of the body will actually increase during crack extension. In this case the energy requirement for crack extension is met from the work done by the external agency, less the increase in the body's strain energy. Hence, $G = \frac{\partial}{\partial A}\left(W.D. - U\right)$. For the general case, with mixed types of loading, a potential energy is defined: $P = U - \sum_i F_i D_i$, where F_i are the loads which are constant during crack extension (i.e. the load-controlled loads) and D_i are the displacements at their point of application. The extension to distributed loads, e.g. pressure, is obvious. Hence, the general definition of energy release rate is $G = -\frac{\partial P}{\partial A}$.

Unless otherwise stated, the crack extension is usually assumed to be "self similar". In other words, a straight crack is considered and the extension is along the local x –axis, i.e.

parallel to the crack face with $\theta = 0$. Kinked crack extensions at some angle θ are also of physical interest. Unfortunately, they are difficult to treat analytically. However, energy release rates at an arbitrary angle θ are often evaluated in finite element analyses.

Some authors reserve the symbol G for the energy release rate in the LEFM case. But it will be used here to mean the energy release rate in the general case, possibly even for non-proportional loading in plasticity. In the latter case the physical definition of G as the energy requirement for unit increase in crack area may still be adopted. However, in expression expressions like $G = \frac{\partial}{\partial A}\left(W.D. - U\right)$, the "strain energy", U, must be re-interpreted as the elastic-plastic work integral, $U = \int\left(\int \sigma_{ij} d\varepsilon_{ij}\right) dV$. This is a load-path dependent quantity in general. One possible load-path is: (a)apply load, and then, (b)allow the crack to extend. This corresponds to the physical definition of G as the energy required for crack extension under load. In non-linear elasticity this would be equivalent to applying the load separately to two bodies with slightly differing crack size, and then considering the difference in their energies. But the equivalence of the two situations fails in the general plastic case.

3. Proof that $J_1 = G$ for Non-Linear Elasticity (Deformation Theory)

Consider a contour, Γ, of the usual type enclosing the crack tip, i.e. starting and ending on the lower and upper crack faces respectively. Suppose the crack suffers a small displacement, $\overline{\Delta}$. We wish to evaluate the energy which flows into the defect during the crack movement. Consider the boundary $\widetilde{\Gamma}$ defined by displacing Γ by $\overline{\Delta}$. The state of stressing and deformation within $\widetilde{\Gamma}$ after the crack extension can be represented in two steps. As a very crude approximation, all the stresses, strains, displacements and energy densities at a point $\bar{r}$ within $\widetilde{\Gamma}$ after crack extension might be equated with those at the point $\bar{r} - \overline{\Delta}$ within Γ before crack extension. This crude approximation leads to a means of analysing the situation exactly.

Suppose that the tractions and displacements on the boundary Γ are initially T and u. And suppose that, after the displacement of the crack, the tractions and displacements on the displaced boundary, $\widetilde{\Gamma}$, are $\widetilde{T}$ and $\tilde{u}$. The correct physical situation can be reached in two notional steps:-

[1] At the initial crack size, consider the region within Γ to be cut out from the body, but holding the tractions on Γ fixed. Consider the region $\widetilde{\Gamma}$ to be cut out of a copy of the body after crack extension. Then the initial cut-out Γ will drop neatly into place in the hole left by the removal of $\widetilde{\Gamma}$ from the second body after the crack displacement[1].

[2] But after step (1) there will be a traction mismatch over the boundary, because T does not equal $\widetilde{T}$. The second step is therefore to adjust the tractions acting on the material within the boundary so that $T \to \widetilde{T}$, which leads also to $u \to \tilde{u}$.

[1] The alert will suspect that this limits the argument to small displacements. It does not. The cut-out fits precisely, even for large displacements, after Step (2) has been completed.

Step (1) is the initial crude approximation. Step (2) makes this notional process precise.

How much would the strain energy within Γ change if step (1) were accurate? In the approximation of step (1), the strain energy density at point $\bar{r}$ after the crack displacement is just $W(\bar{r}-\bar{\Delta})$, where $W(\bar{r})$ is the strain energy density at the same point before the crack movement. Hence, for step (1), the strain energy change within the boundary Γ is just,

$$\Delta E_1 = \int_V W(\bar{r}-\bar{\Delta})dV - \int_V W(\bar{r})dV \tag{1}$$

where V is the volume within Γ. But,

$$W(\bar{r}-\bar{\Delta}) - W(\bar{r}) = -\bar{\Delta}\cdot\bar{\nabla}W \tag{2}$$

and the "divergence" theorem (here, more properly, the "gradient" theorem) gives us,

$$\int_V \bar{\nabla}W dV = \oiint_\Gamma W d\bar{S} = \oiint_\Gamma W\hat{n}dS \tag{3}$$

noting that the integral is over the vector element of area ($\hat{n}$ being the outward normal). The boundary Γ is here taken to symbolise any "closed" surface in 3-dimensions, whose 2D section in the (x,y) plane is the boundary Γ. The inverted commas around "closed" is to remind us that the surface is not really closed but terminates on the crack faces. Hence, if we take the crack displacement to be in the Cartesian direction $\hat{x}_i$, the strain energy change within the boundary Γ (now a 3D surface) due to step (1) is,

$$\Delta E_1 = -\Delta_i \oiint_\Gamma W n_i dS \tag{4}$$

In step (2) the adjustment of the tractions, or more precisely the corresponding adjustment of the boundary displacements, causes work to be done on the material within Γ. Per unit surface area of $\tilde{\Gamma}$ this work is approximately,

$$\frac{1}{2}\left(\tilde{\bar{T}}+\bar{T}\right)\cdot\left(\tilde{\bar{u}}-\bar{u}\right) \approx \bar{T}\cdot\left(\tilde{\bar{u}}-\bar{u}\right) \tag{5}$$

and this becomes precise in the calculus limit of small crack displacements. But at the end of step (1) we have not achieved the correct displacements $\tilde{\bar{u}}$. The approximation after step (1) is that the displacements are given by the initial field $\bar{u}$, but evaluated at the point $\bar{r}-\bar{\Delta}$. Hence, the displacements which have been assumed up to step (1) are,

$$\bar{u}(\bar{r}-\bar{\Delta}) \approx \bar{u}(\bar{r}) - (\bar{\Delta}\cdot\bar{\nabla})\bar{u} \tag{6}$$

Step (2) therefore applies additional displacements $\tilde{\bar{u}}(\bar{r}) - \bar{u}(\bar{r}) + (\bar{\Delta}\cdot\bar{\nabla})\bar{u}$ to the boundary. This requires a corresponding amount of energy per unit area: $\bar{T}\cdot\left[\tilde{\bar{u}}(\bar{r}) - \bar{u}(\bar{r}) + (\bar{\Delta}\cdot\bar{\nabla})\bar{u}\right]$. This is partly a simulation of the work done by rest of the body on the material within Γ, as given by Equ.(5). The balance is left over to increase the strain energy within Γ. The net effect is that step (2) causes the strain energy within Γ to increase by,

$$E_2 = \oint\!\!\!\!\!\!\int_\Gamma \bar{T}\cdot\left[(\bar{\Delta}\cdot\bar{\nabla})\bar{u}\right]dS = \Delta_i \oint\!\!\!\!\!\!\int_\Gamma \bar{T}\cdot\frac{\partial\bar{u}}{\partial x_i}dS \qquad (7)$$

Overall, the total strain energy increase within Γ, adding (4) and (7), is,

$$-\Delta_i \oint\!\!\!\!\!\!\int_\Gamma \left[Wn_i - \bar{T}\cdot\frac{\partial\bar{u}}{\partial x_i}\right]dS \qquad (8)$$

But the energy release rate is minus this energy change divided by the crack displacement. Hence,

$$G_i = \oint\!\!\!\!\!\!\int_\Gamma \left[Wn_i - \bar{T}\cdot\frac{\partial\bar{u}}{\partial x_i}\right]dS \equiv J_i \qquad (9)$$

where G_i represents the energy release rate for crack *displacement* in the Cartesian direction $\hat{x}_i$. Equ.(9) shows that the expression for G_i is identical to that for the vector integrals, J_i.

In particular, for self-similar crack extension along the local x-direction (i.e. for $\theta = 0$), we have $J_1 = G_1$, usually written simply as $J = G$.

4. So What Do J_2 and J_3 Mean? Two Warnings

The first warning is that, whilst J is *defined* by the integral (9) even for plastic behaviour and non-proportional loading, G only equals this integral definition for non-linear elasticity (or plasticity for proportional, monotonic, loading). The general definition of G is, of course, the energy release rate. So J = G ceases to apply in general plasticity, e.g. when unloading of parts of the yielded material has occurred.

The second warning relates to the meaning of J_2 and J_3, which are also defined by the integral, (9). The derivation of Section 3 now makes clear what these quantities mean physically. The key lies in the fact that the whole of the material within the boundary Γ is displaced uniformly in the derivation. This means that the crack itself is displaced bodily. In the case of a displacement in the local x-direction, this is equivalent to a crack extension. But in the case of, say, a displacement in the y-direction, it means that the whole crack is displaced sideways to a different y-coordinate. So for anything other than x-direction displacements, it is not just the crack tip that moves, but the whole crack. This means that J_2 and J_3 (or G_2 and G_3, which are the same thing for non-linear elasticity), are

not the energy release rates for growth of a crack in a kinked direction, as might naively have been imagined.

Since bodily movement of a crack sideways does not correspond to a physically realistic change in the state of a body, the J_2 and J_3 integrals are of little interest in fracture mechanics. They could still be used as a means of mode separation, but alternative accurate methods exist in finite element codes. There may be greater interest in these quantities in the theory of lattice defects, for which the 'sideways' movement may be physically realisable (e.g. dislocation glide).

5. LEFM Expression for Energy Release Rate ($\theta = 0$)

The LEFM fields show that the stresses directly ahead of a crack tip (on $\theta = 0$) under mixed mode loading are,

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}, \quad \tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}, \quad \tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \tag{10}$$

The displacements of the upper crack face ($\theta = \pi$) are,

$$u_y = 4\kappa \frac{K_I}{E}\sqrt{\frac{r}{2\pi}}, \qquad u_x = 4\kappa \frac{K_{II}}{E}\sqrt{\frac{r}{2\pi}}, \qquad u_z = 4(1+\nu)\frac{K_{III}}{E}\sqrt{\frac{r}{2\pi}} \tag{11}$$

Imagine tractions to be applied to the crack faces near the tip in order to cause the crack to close over a length Δ from the original tip. Clearly the tractions are simply given in terms of the stresses, (10), ahead of the new (smaller) crack, at a distance $r' = \Delta - r$. As long as the crack closure is infinitesimal, the SIFs can be taken as unchanged. Assuming elastic behaviour, the work done by the tractions in closing a length δr of the crack tip region is simply the half the product of the corresponding tractions and displacements:-

$$\delta WD = \frac{t\delta r}{2}\left\{\sigma_y(r')u_y(r) + \tau_{xy}(r')u_x(r) + \tau_{yz}(r')u_z(r)\right\} \times 2 \tag{12}$$

where the factor of 2 accounts for the two crack faces, and t is the thickness in the crack front direction (z). Substituting (10, 11) and integrating to find the total work done to close a finite length Δ gives,

$$WD = t\int_0^{\Delta} dr'\left\{4\kappa\frac{K_I^2}{2\pi E}\sqrt{\frac{r}{r'}} + 4\kappa\frac{K_{II}^2}{2\pi E}\sqrt{\frac{r}{r'}} + 4(1+\nu)\frac{K_{III}^2}{2\pi E}\sqrt{\frac{r}{r'}}\right\} \tag{13}$$

The required integral can be evaluated using the substitution $r' = \Delta\cos^2\theta$. This gives,

$$\int_0^{\Delta}\sqrt{\frac{r}{r'}}dr' = \frac{\pi}{2}\Delta \tag{14}$$

So that (13) becomes simply,

$$\frac{WD}{t\Delta} = \kappa\frac{K_I^2}{E} + \kappa\frac{K_{II}^2}{E} + (1+\nu)\frac{K_{III}^2}{E} \qquad (15)$$

which is just the familiar expression for energy release rate G = WD/tΔ. (NB: $\kappa = 1$ in plane stress and $\kappa = 1-\nu^2$ in plane strain).

6. J for a Moving Crack (Non-Linear Elasticity)

Finally, as a purely pedagogic exercise, we can considered a dynamic situation in which kinetic energies enter the problem. The value of this is to reinforce the interpretation of J as the energy flowing into the crack tip (in the case of deformation theory).

The derivation uses the Noether's Theorem method, as described in a separate Note. The Lagrangian is now,

$$L = W\left(u_i, u_{i,j}\right) - \frac{1}{2}\rho\left|\frac{\partial \bar{u}}{\partial t}\right|^2 \qquad (16)$$

where W is the usual strain energy and ρ is the material density. The second term is the kinetic energy density. Using a convention in which Latin letters stand for the spatial degrees of freedom [x, y, z or 1, 2, 3] but Greek letters stand for space or time degrees of freedom [t, x, y, z or 0, 1, 2, 3], the Euler-Lagrange equations now become,

$$\partial_\mu\left(\frac{\partial L}{\partial u_{i,\mu}}\right) = \frac{\partial L}{\partial u_i} \qquad (17)$$

which gives:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + b_i \qquad (18)$$

which is the usual equation of motion, both sides being numerically equal to the net force on a unit volume of material. For a homogeneous medium, the Lagrangian has no explicit spatial dependence, and the total spatial derivative is given by,

$$\frac{dL}{dx_i} = \frac{\partial L}{\partial u_j}\cdot\frac{\partial u_j}{\partial x_i} + \frac{\partial L}{\partial u_{j,\mu}}\cdot\frac{\partial u_{j,\mu}}{\partial x_i} = \partial_\mu\left(\frac{\partial L}{\partial u_{j,\mu}}\right)\cdot\frac{\partial u_j}{\partial x_i} + \frac{\partial L}{\partial u_{j,\mu}}\cdot\frac{\partial u_{j,\mu}}{\partial x_i} = \partial_\mu\left[\frac{\partial L}{\partial u_{j,\mu}}\cdot\frac{\partial u_j}{\partial x_i}\right] \qquad (19)$$

where (17) has been used. Equ.(19) can be re-arranged in the form,

$$T_{i\mu,\mu} = 0 \qquad \text{where, } T_{i\mu} = L\delta_{i\mu} - \frac{\partial L}{\partial u_{j,\mu}}\cdot\frac{\partial u_j}{\partial x_i} \qquad (20)$$

The usual J integral is defined from the spatial parts of this T quantity,

$$J_i = \int T_{ik,k}\,dV = \oint\!\!\int T_{ik}\,dS_k \tag{21}$$

where, to get the usual J, the "closed" contour is replaced by a contour which starts and ends on opposite crack faces. But Equ.(20) tells us that,

$$\int T_{ik,k}\,dV = -\int \frac{\partial T_{i0}}{\partial t}\,dV \tag{22}$$

and also that,

$$T_{i0} = -\frac{\partial L}{\partial u_{j,0}} \cdot \frac{\partial u_j}{\partial x_i} = \rho \frac{\partial u_j}{\partial t} \cdot \frac{\partial u_j}{\partial x_i} \tag{23}$$

Hence,

$$\frac{\partial T_{i0}}{\partial t} = \rho \frac{\partial^2 u_j}{\partial t^2} \cdot \frac{\partial u_j}{\partial x_i} + \frac{1}{2}\rho \frac{\partial}{\partial x_i}\left(\left| \frac{\partial \bar{u}}{\partial t} \right|^2 \right) \tag{24}$$

The simplest interpretation is to assume initially that accelerations are small, and hence that the second term in (24) dominates. This second term is just the gradient of the kinetic energy density, and hence we find,

$$J_i = \int T_{ik,k}\,dV = -\int \frac{1}{2}\rho \frac{\partial}{\partial x_i}\left(\left| \frac{\partial \bar{u}}{\partial t} \right|^2 \right) dV = -\oint\!\!\int \frac{1}{2}\rho \left(\left| \frac{\partial \bar{u}}{\partial t} \right|^2 \right) dS_i \tag{25}$$

Thus, for a notional crack displacement by $\bar{\Delta}$,

$$\bar{J} \cdot \bar{\Delta} = -\oint\!\!\int \frac{1}{2}\rho \left(\left| \frac{\partial \bar{u}}{\partial t} \right|^2 \right) d\left(\bar{\Delta} \cdot \bar{S} \right) \tag{26}$$

But the integration element $d\left(\bar{\Delta} \cdot \bar{S}\right)$ is just the signed volume element between the contours Γ and $\tilde{\Gamma}$. Thus, (26) is just the kinetic energy equivalent of Equ.(4). It is the amount by which the kinetic energy within Γ is increased due to the crack displacement, considering "Step 1" of the method in Section 3.

Similarly we now see that the acceleration term in (24) is the equivalent of (7), i.e. it is the increase in strain-plus-kinetic energy within Γ due to "Step 2" of the method, i.e. due to matching tractions. This follows because the density times the acceleration gives the dynamic component of the body force per unit volume.