Formulation of the Continuum Linear Elasticity Problem in 3D and Compatibility (Small Strain / Small Displacement Theory)

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This is a big subject. The material can all be found in standard texts. However it may be useful to gather together some of the key equations in order to emphasise how they combine to form a soluble system.

The most general linear elasticity problem, in the limit of small displacements, is formulated by three sets of equations, which respectively express equilibrium, the stress-strain relation, and the definition of strain in terms of displacements:-

(A)Equilibrium: $\sigma_{ij,j} = b_i$ (b_i is the applied force per unit volume) (1)

(B)Hooke's Law: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ (generally anisotropic) (2)

(C)Definition of Strain: $\varepsilon_{i,j} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ (3)

In 3D there are three equations of equilibrium but six independent stress components. Clearly, equilibrium alone does not permit the stresses to be found ("statically indeterminate"). The manner in which the material deforms under stress is also required to solve the problem. This is codified by a generalised "Hooke's Law", Equ.(2), which consists of six equations in 3D. However, from the mathematical point of view this gets us no nearer to a soluble system because it also introduces six new unknown quantities, the strain components. To derive a soluble system of equations it is essential to appeal to the definition of the strains in terms of displacements, Equ.(3).

The simplest means of expressing the problem is in terms of the displacements. Substituting (3) into (2) gives the stresses in terms of the displacement gradients. Substituting these expressions into (1) will therefore give us three simultaneous second order differential equations in the three displacement functions. Three equations in three unknowns thus provides a soluble system. Working through this process in the case of an isotropic material gives,

$$\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)u_{x,xx} + \nu u_{y,yx} + \nu u_{z,zx}\right] + \frac{E}{2(1+\nu)}\left[u_{x,yy} + u_{y,xy} + u_{x,zz} + u_{z,xz}\right] = b_x \quad (4)$$

and two similar equations for the y and z directions. It is worth pausing to see what it is that finite element analysis does for us in this context. We can trivially re-write Equ.(4) in terms of a matrix of differential operators, the first row of which is,

$$\hat{D}_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\hat{i}\partial_{xx} + \nu\hat{j}\partial_{yx} + \nu\hat{k}\partial_{zx}\right] + \frac{E}{2(1+\nu)}\left[\hat{i}\partial_{yy} + \hat{j}\partial_{xy} + \hat{i}\partial_{zz} + \hat{k}\partial_{xz}\right] \quad (5)$$

So that (4) becomes: $$\left(\hat{D}\right)\bar{u} = \bar{b} \qquad (6)$$

This compares with the finite element formulation in terms of a stiffness matrix, K, which is $(K)\bar{u} = \bar{b}$. So, we see that the FE stiffness matrix is really just a discrete-space approximation to the second order differential operator defined by (5).

So far the dreaded "compatibility equations" have not appeared. This is because we chose to formulate the problem directly in terms of displacements. If we want to keep the displacements out of the problem, and formulate it in terms of stresses and strains alone then we have a difficulty. There are 9 equations (1) and (2), but 12 unknowns (six stress components and six strain components). The reason for this shortfall is that Equs.(3) express the six unknown strains in terms of just three displacement functions, effectively giving us the missing three equations. If we are to avoid using Equs.(3) then we must use something else which is equivalent but expressed in terms of strains only. The "something else" is the compatibility relations. These are the second order differential relations which must be obeyed by the strains in order for a compatible set of displacements to exist. For an arbitrary set of strain functions, there might not exist any displacement functions for which Equs.(3) would hold. These would not be physically possible strains. To ensure that the strains are physically admissible in this sense can be shown to require that the following six equations hold (in 3D),

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy}, \quad \varepsilon_{xx,zz} + \varepsilon_{zz,xx} = 2\varepsilon_{xz,xz}, \quad \varepsilon_{zz,yy} + \varepsilon_{yy,zz} = 2\varepsilon_{zy,zy} \qquad (7)$$

$$\varepsilon_{xy,xz} + \varepsilon_{xz,xy} = \varepsilon_{yz,xx} + \varepsilon_{xx,yz}, \quad \varepsilon_{yx,yz} + \varepsilon_{yz,yx} = \varepsilon_{xz,yy} + \varepsilon_{yy,xz}, \quad \varepsilon_{zy,zx} + \varepsilon_{zx,zy} = \varepsilon_{yx,zz} + \varepsilon_{zz,yx}$$

Substituting Equs.(3) into any of the Equs.(7) easily shows that they are all identities. The proof that displacement fields obeying Equs.(3) necessarily exist given Equs.(7), i.e. that Equs.(3) are the integrability conditions for Equs.(7), is best left to the mathematicians, but it *is* true. Hence, a complete formulation of the 3D elasticity problem without the explicit appearance of displacements consists of the coupled Equs.(1), (2) and (7).

Incidentally, this now appears to be an *over*-constrained sets of equations, with 12 unknown functions and 15 equations! This illustrates the perils of too naïve an approach to simultaneous differential equations. Whether they are uniquely soluble, and for what boundary conditions, cannot really be determined by simply comparing the number of unknown functions and the number of equations. In this case, the six Equs.(7) only really suffice to reduce six functional degrees of freedom to three, because they are completely equivalent to Equs.(3). Hence, the six Equs.(7) only really provide three extra conditions.

An interesting academic question is, "what are the compatibility equations in other 3D coordinate systems?" A companion Note shows how to derive the compatibility equations in 3D spherical polars. But I doubt that any sane person would want to use these expressions to solve a real problem.