Does Quantum Measurement Violate the Conservation of Angular Momentum?
A Paradox Resolved

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Presented here is a paradox which took some considerable effort to resolve - and I don't think I've got it completely right yet.

1. Spin Measurement

Consider a spin 1/2 particle whose spin is initially prepared as 'spin up' with respect to the z axis (say, vertically upwards). The particle travels horizontally (say along the y axis) into a Stern-Gerlach apparatus which is oriented to measure the x-directed spin of the particle. We know that the outcome will be that half the particles will be measured as 'spin positive' wrt the x-axis, and half will be 'spin negative' wrt the x-axis. This is because, in obvious Hilbert space notation,

$$\left|\uparrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|+\right\rangle_x + \left|-\right\rangle_x\right) \tag{1.1}$$

In matrix notation this can be seen as follows. Work in the z-basis, in which the two states of define z-spin are denoted,

$$\left|\uparrow\right\rangle_z \to \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \left|\downarrow\right\rangle_z \to \begin{pmatrix}0\\1\end{pmatrix} \tag{1.2}$$

in which case the spin operator takes its conventional (Pauli) representation,

$$\hat{S} = \frac{1}{2}\overline{\sigma} = \frac{1}{2}\left\{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right\} \tag{1.3}$$

with coordinates in the usual order: x, y, z. Hence,

$$\hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{1.4}$$

as required, and the states of definite x-spin are,

$$\left|+\right\rangle_x \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \left|-\right\rangle_x \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.5}$$

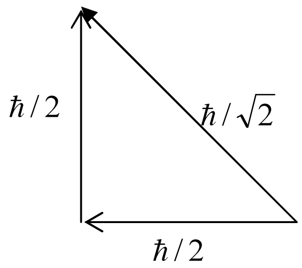
so that, as required,

$$\hat{S}_x\left|+\right\rangle_x = \frac{1}{2}\left|+\right\rangle_x \quad \text{and} \quad \hat{S}_x\left|-\right\rangle_x = -\frac{1}{2}\left|-\right\rangle_x \tag{1.6}$$

and (1.5) is consistent with (1.1).

2. The Paradox

Suppose the outcome of the measurement is the positive x-spin state. The angular momentum of the particle has changed from a magnitude of $\hbar/2$ in the z-direction to the same magnitude in the x-direction. Apparently the only way we can salvage the conservation of angular momentum is by appeal to the measurement apparatus sustaining an equal and opposite change of angular momentum. But the required change in the apparatus's angular momentum is $\hbar/\sqrt{2}$, i.e.,



But this is impossible because angular momentum can be exchanged only in multiples of $\hbar$. So the apparatus cannot balance the change in the particle's angular momentum and it appears that angular momentum conservation is violated. This is the paradox.

After passing through the x-oriented Stern-Gerlach apparatus, the positive x-spin outgoing beam can be passed through a second such apparatus but oriented in the z-direction. The outcome is an equal number of particles emerging spin up and spin down wrt the z-direction, because, of course,

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right) \tag{2.1}$$

However, exactly the same is true if the negative x-spin beam is passed through the second Stern-Gerlach apparatus, because,

$$|-\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z - |\downarrow\rangle_z\right) \tag{2.2}$$

And, using the same reasoning as before, the second apparatus cannot balance the change in the particles' angular momenta. So, overall, the above process appears to have reversed the spins of half the particles without any change to the angular momentum of anything else. Hence the angular momentum of the universe has been changed by $N\hbar/2$, where N is the number of particles processed.

The paradox can be presented in an even more virulent form, in which *all* the particles reverse their spin. Consider a Stern-Gerlach apparatus oriented with its axis in the x, z plane at an angle θ wrt the z-axis. The eigenvectors of the spin operator $\hat{S}_n = \frac{1}{2}\overline{\sigma} \cdot \hat{n}$ where $\hat{n}$ is the unit vector of the apparatus's axis are easily shown to be,

$$|+\rangle_\theta = \cos\frac{\theta}{2}|\uparrow\rangle_z + \sin\frac{\theta}{2}|\downarrow\rangle_z \tag{2.3a}$$

$$|-\rangle_\theta = \sin\frac{\theta}{2}|\uparrow\rangle_z - \cos\frac{\theta}{2}|\downarrow\rangle_z \tag{2.3b}$$

Hence,

$$|\uparrow\rangle_z = \cos\frac{\theta}{2}|+\rangle_\theta + \sin\frac{\theta}{2}|-\rangle_\theta \tag{2.4a}$$

$$|\downarrow\rangle_z = \sin\frac{\theta}{2}|+\rangle_\theta - \cos\frac{\theta}{2}|-\rangle_\theta \tag{2.4b}$$

Consider the Stern-Gerlach apparatus to be at some very small angle $\delta\theta = \pi / m$ for some very large integer m. The probability that the particle, initially spin up wrt the z-axis, emerges as having positive spin in the θ-direction is,

$$P = \cos^2 \frac{\theta}{2} \approx 1 - \left(\frac{\pi}{2m}\right)^2 \quad (2.5)$$

The change in the particle's angular momentum in this case is $\hbar \sin \delta\theta / 2 =$ $\hbar \sin \pi / 2m$. This reinforces the apparent impossibility of the apparatus balancing this change in angular momentum, because, for large m, it is very small indeed - and hence well below the quantum of exchange, $\hbar$.

If we now imagine a succession of m Stern-Gerlach devices, each at the same angular spacing, $\delta\theta$, so the axis of the last apparatus is oriented vertically downwards, we find that the probability of the particle emerging with downward pointing (i.e., negative z) spin is,

$$P^m \approx \left[1 - \left(\frac{\pi}{2m}\right)^2\right]^m \to 1 \text{ as } m \to \infty \quad (2.6)$$

So, this arrangement of many devices allows us to gradually coax the particle's spin to turn around to point downwards, apparently without any balancing change in the angular momentum of anything else. Disaster for the conservation of angular momentum!

3. The Paradox is All About Quantum Measurement

The paradox arises due to the nature of the quantum measurement process. Consider the original version with just one, x-oriented, Stern-Gerlach device. We have tacitly assumed that passage through this device constitutes a measurement. In standard QM measurement theory this means that the initial state $\left|\uparrow\right\rangle_z = \left(\left|+\right\rangle_x + \left|-\right\rangle_x\right)/\sqrt{2}$ undergoes "collapse of the wavefunction" and emerges as either $\left|+\right\rangle_x$ or as $\left|-\right\rangle_x$ with equal probability. In view of $\left|+\right\rangle_x = \left(\left|\uparrow\right\rangle_z + \left|\downarrow\right\rangle_z\right)/\sqrt{2}$, the former case will subsequently be measured by a z-oriented apparatus as spin up or spin down with equal probability - and the same is true for the case $\left|-\right\rangle_x$. But the collapse of the wavefunction into either $\left|+\right\rangle_x$ or $\left|-\right\rangle_x$ by the x-oriented device is crucial to this reasoning.

There is an alternative experimental arrangement, which is identical to that described except that we assume that there is no way of distinguishing between the states $\left|+\right\rangle_x$ and $\left|-\right\rangle_x$ emerging form the x-oriented device. A real Stern-Gerlach apparatus separates these two states spatially, one beam being deflected up and the other down wrt the magnetic axis. However, it is still necessary to detect which path is followed in order to infer the result of the spin measurement. Suppose the experimental set-up includes no means of detecting the path followed (think interferometer!). In this case, the initial state $\left|\uparrow\right\rangle_z = \left(\left|+\right\rangle_x + \left|-\right\rangle_x\right)/\sqrt{2}$ undergoes no collapse. It remains in a coherent superposition of x-spin states - namely that particular superposition which is identical to a z-spin-up state, of course. So, when the particle passes through a second, z-oriented, Stern Gerlach device, the result is a measurement of z-spin up 100% of the time. No angular momentum change, and no paradox.

The same conclusion applies to the case of m devices at relative angles of π / m. If these devices do not actually constitute a measurement, the state never changes. In the formulation of the paradox, however, it has been assumed that every one of these devices carries out a measurement - with the associated wavefunction collapse - since we have blithely talked about the "probability" of being in a specified state. But without wavefunction collapse we can only refer to coherent probability amplitudes, not probabilities.

The paradox as formulated therefore hinges crucially on the nature of quantum measurement. However, there ***is*** still a paradox which needs resolution because, if we *do* allow the Stern-Gerlach devices to provide a genuine measurement then the angular momentum of the particle can indeed end up being reversed.

4. Quantum Measurement - Decoherence Theory

This Section is not crucial to the resolution of the paradox - skip it if you wish.

The simplistic view of quantum measurement is as follows. Suppose we wish to measure the quantity represented by Hermitian operator $\hat{Q}$ and the state of the system is,

$$|\psi\rangle = \alpha|q_1\rangle + \beta|q_2\rangle \qquad (4.1)$$

where $|\alpha|^2 + |\beta|^2 = 1$ and where $|q_1\rangle, |q_2\rangle$ are eigenstates of $\hat{Q}$ with eigenvalues q_1, q_2 respectively. (The generalisation to any number of contributing eigenstates is obvious). A measurement of $\hat{Q}$ resulting in the outcome q_1 is considered to 'collapse the wavefunction' to leave the system in the state $|q_1\rangle$. According to the Born Rule this occurs with probability $|\alpha|^2$.

This 'collapse the wavefunction' business has been the bugbear of quantum mechanics from the start. Schrodinger never liked it at all. It introduces into quantum mechanics a special sort of temporal evolution which applies only during those special events which are labelled 'measurements'. The rest of the time, quantum systems evolve unitarily according to the Schrodinger equation, i.e., $\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$. To my knowledge, there has been no fully satisfactory resolution of the quantum measurement conundrum as yet. However, there is a partial clarification of what happens in quantum measurement known as 'decoherence theory'. For an explanation of decoherence theory see Zurek, for example, or my own exposition. However, the essence of it is simple.

To model a measurement as a decoherence process it is essential to include the measurement apparatus in the state. Let's say the apparatus starts in state $|A_0\rangle$. Prior to interaction with the system, the combined state is the simple direct product $|A_0\rangle|\psi\rangle$. In order for the apparatus to carry out a measurement it is necessary for the apparatus to physically interact with the system. Moreover, it is essential that the apparatus change its state - otherwise how could we distinguish between one measurement outcome and another? The interaction between the apparatus and the system obeys standard quantum mechanics, the evolution being determined by the (unitary) Schrodinger equation describing their interaction according to some interaction Hamiltonian, $\hat{H}_I$. If $|\Psi\rangle$ is the combined state of the apparatus-plus-system then the temporal evolution is controlled by $\hat{H}|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$ where $\hat{H}$ is the sum of the free-system and free-apparatus Hamiltonians and the interaction Hamiltonian, $\hat{H}_I$. As a result the initial state, $|A_0\rangle|\psi\rangle$, evolves as follows,

$$|A_0\rangle|\psi\rangle = |A_0\rangle[\alpha|q_1\rangle + \beta|q_2\rangle] \to |\Psi\rangle = \alpha|A_1\rangle|q_1\rangle + \beta|A_2\rangle|q_2\rangle \qquad (4.2)$$

where $|A_1\rangle$ is the state of the apparatus which we can recognise as indicating the measurement outcome q_1 (e.g., the needle on the dial swings to "q_1"), and similarly for $|A_2\rangle$. So far, so unitary.

At this point it is crucial to point out that the most general state of a system is ***not*** representable by a ray in Hilbert space. Such systems are said to be in a pure quantum state. The error of assuming any physical system can be described by a ray in Hilbert space is what leads to paradoxes like Schrodinger's cat. On the basis of this error we are led to believe that a cat can be represented by a pure quantum state, and hence that, since superpositions of Hilbert space states constitute another valid state, it makes sense to talk about a state defined as a superposition of a living cat and a dead cat. It is not so - because the initial premise is false: a living cat cannot be described by a pure quantum state. The very nature of "life" precludes it.

Instead, the most general state of any physical system is describable by a density matrix. It is not necessary to address the formulation of the state of the most general system here - see the first couple of Chapters of my introduction to quantum mechanics. It suffices for our present purposes that, in the special case of a pure quantum state, $|\Psi\rangle$, the so-called density matrix is the Hilbert space operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$. When this state is the combined state of a system-plus-measuring-apparatus, decoherence theory says that the state of the system (alone) which results from the measurement is obtained by averaging over all states of the measuring apparatus. This involves taking the matrix element of the density matrix between every apparatus state, and summing over them, thus,

$$\hat{\rho}_{red} = \sum_{A_i} \langle A_i|\Psi\rangle\langle\Psi|A_i\rangle \qquad (4.3a)$$

where the so-called "reduced" density matrix, $\hat{\rho}_{red}$, describes the system only - the apparatus having vanished from the prescription by virtue of being averaged-over. The operation in (4.3) is referred to as "tracing out the apparatus".

A key requirement of a good measuring device is that the states $|A_i\rangle$ are definitely distinguishable for different i. This means that they must be orthogonal states, i.e.,

$$\langle A_i|A_j\rangle = \delta_{ij} \qquad (4.3b)$$

Applying (4.3a) and (4.3b) to (4.2) gives,

$$\hat{\rho}_{red} = |\alpha|^2|q_1\rangle\langle q_1| + |\beta|^2|q_2\rangle\langle q_2| \qquad (4.4)$$

This is a purely classical (i.e., stochastic) mixture of the pure states $|q_1\rangle$ and $|q_2\rangle$. It is *not* a superposition, unlike (4.1). Nor is it a pure quantum state. Instead (4.4) means that you will find, as a result of the measurement, that the system is in state $|q_1\rangle$ with relative frequency $|\alpha|^2$ and in state $|q_2\rangle$ with relative frequency $|\beta|^2 = 1-|\alpha|^2$. However, decoherence theory does not fully explain the mysteries of quantum measurement because it is still the case that the system after measurement 'collapses' into one of the two pure states, $|q_1\rangle$ or $|q_2\rangle$.

The important point, however, is that the measurement process - as represented by the tracing-out in (4.3) - changes the state of the system as given by the density matrix. After the measurement (4.4) is the system's density matrix, whereas before the measurement it was,

$$\begin{aligned} |\psi\rangle\langle\psi| &= [\alpha|q_1\rangle + \beta|q_2\rangle][\alpha^*\langle q_1| + \beta^*\langle q_2|] \\ &= |\alpha|^2|q_1\rangle\langle q_1| + |\beta|^2|q_2\rangle\langle q_2| + \alpha\beta^*|q_1\rangle\langle q_2| + \alpha^*\beta|q_2\rangle\langle q_1| \end{aligned} \quad (4.5)$$

The difference between the mixed state, (4.4), and the pure state, (4.5), lies in the presence of the cross-terms in the latter. In general, these cross terms are responsible for interference effects. Decoherence causes the cross-terms (or off-diagonal terms) to vanish. They do not survive tracing-out the apparatus, (4.3) (or, more generally, tracing out the environment).

5. Resolution of the Paradox

The resolution of the paradox presented in §2 is that the measuring apparatus can, in fact, change its angular momentum state. §2 is perfectly correct as regards the changes in the state of the particle. The particle's spin state can indeed reverse. But if so, the angular momentum state of the apparatus also changes to keep the combined angular momentum unchanged. The challenge, then, is to understand why the argument in §2, namely that this would involve impossible changes in angular momentum of fractions of $\hbar$, is invalid.

We can know with precision only one coordinate component of the total angular momentum. Lets say it is the z-component, and call it J_z (in $\hbar$ units). This can be expressed as the sum of the spin of the particle and the combined angular momentum of the apparatus and the relative orbital angular momentum of the particle and the apparatus. The latter must be some multiple L_z of $\hbar$ where L_z is an integer or half-integer. Hence,

$$J_z = L_z + \frac{1}{2} \quad (5.1)$$

because the particle is initially in the z-spin-up state. This total angular momentum must remain unchanged when the particle passes through the Stern-Gerlach device provided that the device-plus-system is isolated from any other influences. It is clear, though, that the spin of the particle could be reversed if the angular momentum of the apparatus were increased to $L_z + 1$, so that the total angular momentum would be unchanged, i.e., symbolically $J_z = (L_z + 1) - \frac{1}{2}$. The most general state of the particle-plus-apparatus is thus,

$$|\psi\rangle = \alpha|L_z\rangle|\uparrow\rangle_z + \beta|L_z+1\rangle|\downarrow\rangle_z \quad (5.2)$$

noting that this is an eigenstate of $\hat{J}_z$ with eigenvalue $J_z = L_z + 1/2$. Re-writing this in terms of x-spin states using (1.1) and $|\downarrow\rangle_z = (|+\rangle_x - |-\rangle_x)/\sqrt{2}$ gives, after re-arranging,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{(\alpha|L_z\rangle + \beta|L_z+1\rangle)|+\rangle_x + (\alpha|L_z\rangle - \beta|L_z+1\rangle)|-\rangle_x\} \quad (5.3a)$$

If the Stern-Gerlach apparatus is to constitute a reliable measuring device for the x-spin, it follows that the two apparatus states appearing in (5.3), namely, $(\alpha|L_z\rangle + \beta|L_z+1\rangle)$ and $(\alpha|L_z\rangle - \beta|L_z+1\rangle)$, must be orthogonal (see §4). Hence we may take $\alpha = \beta = 1/\sqrt{2}$ and (5.3a) becomes,

$$|\psi\rangle = \frac{1}{2}\{(|L_z\rangle + |L_z+1\rangle)|+\rangle_x + (|L_z\rangle - |L_z+1\rangle)|-\rangle_x\} \quad (5.3b)$$

This already hints at the resolution of the paradox. The particle state $|+\rangle_x$ is now paired with the apparatus state $(|L_z\rangle + |L_z+1\rangle)/\sqrt{2}$ which does not have a definite z-component of angular momentum. Roughly speaking, this is how quantum mechanics gets round the problem that the apparatus cannot change its angular momentum by a fraction of $\hbar$. The nearest it can get to this is to become a superposition of an unchanged state and a state which has changed angular momentum by one $\hbar$ unit.

Suppose the outcome of the measurement is the particle state $|+\rangle_x$. There are two ways of thinking about what state the apparatus is left in. Strictly, according to decoherence theory (see §4), if the particle-plus-Stern-Gerlach are isolated, and the latter is taken to be a measurement device, then the result of the measurement is obtained by averaging over all states of the apparatus. From this perspective we strictly cannot say what state the apparatus is left in. However, this is rather unsatisfactory because we have a suspicion that the apparatus changes its angular momentum and so we need to know its state to confirm that angular momentum overall is conserved. So, we image there is a third device present which can 'measure' the Stern-Gerlach. Exactly what it measures depends upon the eigenstate structure of the Hermitian operator which represents the measurement. Because we are only concerned about the principle, we can take this measurement to include eigenstates $(|L_z\rangle + |L_z+1\rangle)/\sqrt{2}$ and $(|L_z\rangle - |L_z+1\rangle)/\sqrt{2}$ (noting that their orthogonality is again crucial to this interpretation). What all this amounts to is that, if the particle is measured to be in state $|+\rangle_x$ we can take (or contrive) the system to be left in state $(|L_z\rangle + |L_z+1\rangle)/\sqrt{2}$, because the product of these states is what appears in (5.3b). Hence, the 'collapsed' combined state of particle-plus-apparatus is,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{(|L_z\rangle + |L_z+1\rangle)|+\rangle_x\} \quad (5.4)$$

Note that this is not an eigenstate of $\hat{J}_z$. This is hardly surprising because we know we can only measure one of the coordinate components of an angular momentum operator at a time, and the measurement of the spin in the x-direction naturally causes the state to cease to be an eigenstate of the total z angular momentum.

However, if angular momentum is to be conserved on average, the expectation value of $\hat{J}_z$ in the state (5.4) should equal the original value of $J_z = L_z + 1/2$. We have,

(5.5)

$$\hat{J}_z|\psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{L}_z + \frac{1}{2}\sigma_z\right)\left[|L_z\rangle + |L_z + 1\rangle\right]|+\rangle_x$$
$$= \frac{1}{2}\left(\hat{L}_z + \frac{1}{2}\sigma_z\right)\left[|L_z\rangle + |L_z + 1\rangle\right]\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$$
$$= \frac{1}{2}\left[\left(L_z + \frac{1}{2}\right)|L_z\rangle|\uparrow\rangle_z + \left(L_z - \frac{1}{2}\right)|L_z\rangle|\downarrow\rangle_z + \left(L_z + \frac{3}{2}\right)|L_z + 1\rangle|\uparrow\rangle_z + \left(L_z + \frac{1}{2}\right)|L_z + 1\rangle|\downarrow\rangle_z\right]$$

Hence, using $\langle\psi| = \frac{1}{2}\left(\langle L_z| + \langle L_z + 1|\right)\left(\langle\uparrow|_z + \langle\downarrow|_z\right)$ we find,

$$\langle\psi|\hat{J}_z|\psi\rangle = \frac{1}{4}\left[\left(L_z + \frac{1}{2}\right) + \left(L_z - \frac{1}{2}\right) + \left(L_z + \frac{3}{2}\right) + \left(L_z + \frac{1}{2}\right)\right] = L_z + \frac{1}{2} = J_z \qquad (5.6)$$

confirming that angular momentum is conserved. <u>**QED**</u>.

However, a disconcerting feature of (5.5) is that it appears to imply that angular momentum is not conserved for every single particle which passes through the process. (5.5) implies that 25% of cases result in an increase in total angular momentum by one $\hbar$ unit, and also that 25% of cases result in a decrease in total angular momentum by one $\hbar$ unit. So I suspect that the above analysis is not quite complete. My suspicion is that the "third measuring device" alluded to above balances the angular momentum in these cases. A demonstration of this is required.